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PLANE TRIGONOMETRY.

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AN ELEMENTARY TREATISE

ON

PLANE TRIGONOMETRY;

WITH

A Numerous Collection of Examples,

CHIEFLY DESIGNED FOR THE USE OF SCHOOLS AND BEGINNERS.

BY

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PREFACE.

THE following pages are specially intended for use in Schools. In the choice of matter I have been chiefly guided by the requirements of the three days' Examination at Cambridge, with the exception of proportional parts in logarithms which I have omitted. In one point, I have ventured to deviate from the usual custom. I have denoted angles throughout by the Greek characters, even before the explanation of circular measure. My reasons for doing so were chiefly two; first, that they are much more distinctly and easily written, and it is advisable that a boy should be accustomed in the text to the notation he uses on his paper; secondly, I am thus enabled to insert the algebraical symbols for the angles in the figures, a very great advantage in such propositions as that proved in Art. 27. I have added about 400 Examples mainly collected from the Examination Papers of the last ten years, and I have taken great pains to exclude from the body of the work any which might dishearten a beginner by their difficulty.

In conclusion, I must express my great obligations to my friend the Rev. R. B. Mayor, of Rugby. Without his kind encouragement I should not have ventured to have offered these pages to the public; and I am indebted to him, not only for many valuable suggestions, but for a careful revision of the whole work.

R. D. BEASLEY.

ST JOHN'S COLLEGE, CAMBRIDGE,
September, 1858.

ADVERTISEMENT TO THE SECOND EDITION.

The chief alterations in this Edition are the addition of Articles 63, bis, and 73—75, and of the easy series of Examples marked A.

Grammar School, Grantham, *January*, 1865.

ADVERTISEMENT TO THE THIRD EDITION.

In this Edition the Examples have been largely increased, Articles 76—78 have been added, and the series of Examination Papers.

Grammar School, Grantham, *February*, 1872.

ADVERTISEMENT TO THE EIGHTH EDITION.

In this Edition Section IX. on proportional parts has been added, and the number of Examination Papers considerably increased.

Bournemouth, *March*, 1883.

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PLANE TRIGONOMETRY.

SECTION I.

UNITS OF MEASUREMENT. USE OF SIGNS + AND −. MEANING OF THE TERM "ANGLE" IN TRIGONOMETRY. THE TRIGONOMETRICAL RATIOS. PRACTICAL APPLICATION. INSTRUMENTS FOR SURVEYING.

1. *Object of Trigonometry.* In Trigonometry we apply Algebraical symbols to establish certain relations between the magnitudes of the sides and angles of plane rectilineal figures. These relations are useful for all the higher branches of Mathematics, and are specially applicable to surveying, and to determining the heights and distances of inaccessible objects. In the present treatise we shall confine ourselves to the simpler relations, and some practical applications of them.

We must first consider the mode of estimating algebraically the magnitudes of lines, areas, and angles.

2. *Measurement of lines.* As lines have neither breadth nor thickness we have only to measure their length. To do this we take some standard length, as one foot, one inch, five inches, or any other definite length as our unit of measurement; and the length of any line is then measured and represented by the number, whether whole or fractional, of these units which it contains. Thus if 5 inches is our unit, a line of 20 inches is measured by the number 4, and is called the line 4, that is, the line whose length is 4 times the unit of length. So also the line a would be the line whose length is a times the unit of length.

In investigations involving only algebraical symbols it is indifferent what unit is employed: but we must be careful to remember that throughout the same investigation we are using the same unit.

Thus if two lines a, b enter into our calculations we must consider the length of the line a to be a times *some* unit of length, and of the line b to be b times the *same* unit. When we apply our general results to numerical examples we must be careful to consider the special unit employed.

3. In the measurement of superficies we take the square of which one side is the unit of length, as the unit of superficies; and then any area is measured by the number of these units it contains.

N. B. The area of a rectangle whose sides contain a, b linear units respectively, contains ab superficial units. Also the areas of a parallelogram and a triangle whose bases are b and altitudes a are ab and $\frac{ab}{2}$ respectively.

4. *Measurement of angles.* Since the idea of a right angle is simple, and all right angles are equal, it is conveniently taken as a standard by which to measure the magnitudes of other angles. But since most of the angles we have to deal with are less than right angles, and would therefore be represented by fractions or decimals, it is found convenient to divide it.

In England we divide the right angle into 90 equal parts called degrees; each degree is divided into 60 minutes; each minute into 60 seconds. Any angle is then measured by the number of degrees, minutes, seconds, and decimal parts of a second it contains; an angle containing 27 degrees, 13 minutes, 24.53 seconds is written thus, $27^\circ, 13', 24''.53$.

In France the decimal system is adopted. A right angle contains 100 grades, a grade 100 minutes, a minute 100 seconds; an angle containing 33 grades, 27 minutes, 45.5 seconds is written thus, $33^g, 27^{\backprime}, 45^{\backprime\backprime}.5$. The advantage of this system is, that any angle can be reduced to the decimal of a grade at once, and *vice versâ*. Thus $33^g, 27^{\backprime}, 45^{\backprime\backprime}.5 = 33^g.27455$, and $27^g.35679 = 27^g, 35^{\backprime}, 67^{\backprime\backprime}.9$.

5. We will now shew how to change our unit from degrees to grades, and *vice versâ*.

Let D be the number of degrees in any angle ACB, G the number of grades in the same angle; then since the number of degrees in two angles must bear to one another the same ratio which the angles themselves do, we have

$$\frac{\text{angle } ACB}{\text{a right angle}} = \frac{D}{90};$$

for a like reason

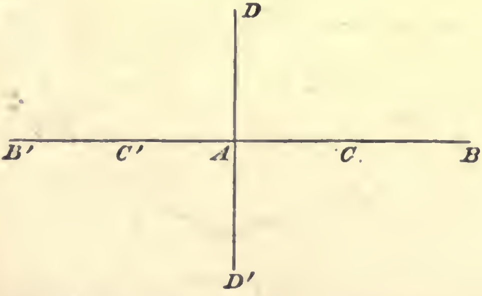
$$\frac{\text{angle } ACB}{\text{a right angle}} = \frac{G}{100}.$$

Hence $\frac{D}{90} = \frac{G}{100}$: from which equation, if we know the number of degrees, we can find the number of grades, and *vice versâ.*

There is also another mode of measuring angles, called *Circular measure.* For an account of this, the student may, if he pleases, turn at once to Section VIII, Articles 65—69, and to the Examples A, appended to that section.

6. *Use of signs + and – to represent contrariety of position.* In applying Algebra to the solution of Arithmetical Problems, we frequently meet with negative results, which require interpretation in each particular case; we shall meet with similar results in Trigonometry; and the question arises, how are we to interpret a negative quantity when it represents a line or angle?

Suppose BAB', DAD' to be two lines at right angles to one another and fixed in position.

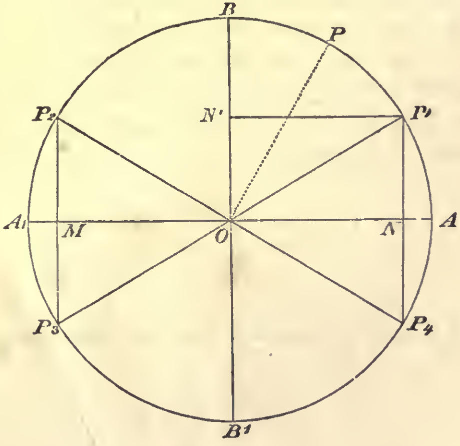


Let $AB = a$, $BC = b$, $AC = x$; then, representing the relation between these lines by Algebraical symbols, we have

$$x = a - b.$$

Now so long as $b < a$, x is positive, and C lies to the right of A: but if $b > a$, x is negative, and C lies to the left of A: that is, in using Algebraical symbols, if we arrive at a negative quantity representing a line, the natural interpretation appears to be, that the line must be considered as measured in an opposite direction to what it would be if the quantity representing it were positive. Hence we are led to the following convention: any line measured along AB or parallel to it is said to be a positive line, and any line measured along AB' or parallel to it is said to be a negative line, and the symbol representing it must have the sign $-$ placed before it. So also lines measured along AD are positive, and along AD' negative. Lines measured in any other direction, not parallel to either of these two, will be considered positive. The advantage of these conventions will appear to the student as he proceeds.

7. *Magnitudes of angles unlimited in Trigonometry.*



In Euclid the magnitude of an angle is the absolute amount of the inclination of the two lines which contain it, to each other; this amount can never exceed two right angles. In Trigonometry the idea of an angle is extended. Suppose AOA' to be a line fixed in space, and OP to be a line revolving from its original position OA towards OB, and tracing out the angle AOP: till OP reaches OA', this angle does not differ from Euclid's idea of an angle, because it is less than two right angles; but when OP has passed OA', and arrived at such a position as OP_3, the Trigonometrical idea of the angle differs from Euclid's: Euclid would now measure the angle AOP_3 by the amount of inclination of OP_3 to OA, that is, he would consider it as less than two right angles; but in Trigonometry we consider it as the angle actually described by the revolving line OP, that is, as $180^\circ + \angle A'OP_3$. So also when the revolving line reaches OP_4, the Trigonometrical angle AOP_4 will be $270^\circ + \angle B'OP_4$; and when OP has passed its initial position OA and reached OP_1 a second time, the Trigonometrical angle AOP_1 will be $360^\circ + \angle AOP_1$. Thus if AOP_1 is 30°, then AOP_2 is 150°, AOP_3 is 210°, AOP_4 is 330°; and in the second revolution, when OP has reached OP_1, the $\angle AOP_1$ is 390°. The same plan for estimating the value of the angle extends to any number of revolutions of OP, and in this sense the magnitude of an angle in Trigonometry is unlimited.

If the line OP on starting from OA had revolved in the opposite direction, towards OB', then in accordance with the principle laid down in Art. 6, we should have considered the angle described as a negative angle; and we have the same convention for angles that we have for lines, viz. that angles measured upwards from OA towards OB are positive angles, and angles measured downwards from OA to OB' are negative angles, and have the sign $-$ placed before the symbol representing them.

The same position of the line OP may be considered as making either a positive or negative angle with OA. Thus when OP is at OP_3, it may be supposed to have revolved either in a positive direction through OB, OA_1, or in a negative direction through OB_1, i.e., AOP_3 may be either 210° or -150°.

Obs. The space between OA and OB is called the first quadrant; when OP lies between OA and OB, the angle AOP is said to be an angle in the first quadrant. So also angles AOP_2, AOP_3, AOP_4 are angles in the second, third, and fourth quadrants respectively.

8. *Complement; Supplement.* The complement of an angle is that angle which must be added to it to make it equal to a right angle. Thus the complement of

$$30^\circ \text{ is } 60^\circ, \text{ of } 120^\circ \text{ is } -30^\circ, \text{ of } -75^\circ \text{ is } 165^\circ.$$

The supplement of an angle is that angle which must be added to it to make it equal to two right angles. Thus the supplement of

$$30^\circ \text{ is } 150^\circ, \text{ of } 220^\circ \text{ is } -40^\circ, \text{ of } -75^\circ \text{ is } 255^\circ.$$

It is evident that in a right-angled triangle one of the acute angles is the complement of the other; and in any triangle any one of the angles is supplementary to the sum of the other two.

9. *Trigonometrical Ratios.* Having now shewn how lines and angles are to be measured in Trigonometry, and the interpretation which is to be given to the signs + and – when applied to quantities that represent them, we will proceed to shew how lines are connected with angles: this is done by means of what are called Trigonometrical Ratios.

Let the revolving line OP (fig. Art. 7) have come into any position OP_1, OP_2, ..., and from P_1, P_2, ..., drop the perpendiculars P_1N, P_2M, ... upon the initial line OA, or OA produced backwards to A'; the triangles P_1NO, P_2MO, ... will be right-angled triangles. We then have the following definitions.

The ratio of the perpendicular to the hypothenuse is called the *sine* of the angle described: that is,

$$\frac{P_1N}{OP_1}, \frac{P_2M}{OP_2}, ...$$

are the sines of the angles AOP_1, AOP_2, ... respectively.

The ratio of the base to the hypothenuse is called the *cosine* of the angle, that is,

$$\frac{ON}{OP_1}, \frac{OM}{OP_2}, ...$$

are the cosines of AOP_1, AOP_2,.... respectively.

The ratio of the perpendicular to the base is called the *tangent* of the angle, that is,

$$\frac{P_1N}{ON},\ \frac{P_2M}{OM},\dots$$

are the tangents of AOP_1, AOP_2, ... respectively.

The ratio of the base to the perpendicular is called the *cotangent* of the angle, that is,

$$\frac{ON}{P_1N},\ \frac{OM}{P_2M},\dots$$

are the cotangents of AOP_1, AOP_2, ... respectively.

The ratio of the hypothenuse to the base is called the *secant* of the angle, that is,

$$\frac{OP_1}{ON},\ \frac{OP_2}{OM},\dots$$

are the secants of AOP_1, AOP_2,... respectively.

The ratio of the hypothenuse to the perpendicular is called the *cosecant* of the angle, that is,

$$\frac{OP_1}{P_1N},\ \frac{OP_2}{P_2M},\dots$$

are the cosecants of AOP_1, AOP_2, ... respectively.

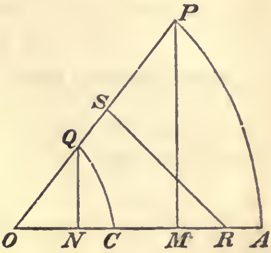
These definitions are quite independent of the direction and number of the revolutions of OP about O; that is, they apply to negative angles, and to angles greater than four right angles.

The Trigonometrical Ratios of any angle a (that is, of any angle which contains a degrees, grades, or any other unit we may employ) are usually written

$$\sin a,\ \cos a,\ \tan a,\ \cot a,\ \sec a,\ \operatorname{cosec} a.$$

10. The question here naturally arises—are these Trigonometrical ratios definite quantities for any given value of the angle, or do they depend upon the magnitude of the revolving line OP?

Let AOP be any angle, i.e. let OP be any position whatever of the revolving line. Take Q any point in OP, or OP produced; draw PM, QN perpendicular to OA.



Then by similar triangles PMO, QNO,

$$QN : ON : OQ = PM : OM : OP.$$

Hence the Trigonometrical Ratios of the angle AOP are the same whether we consider OQ or OP the revolving line.

So also if RS is perpendicular to OP, we have

$$RS : OS : OR = PM : OM : OP.$$

Hence we conclude, that, *disregarding signs*, the values of the Trigonometrical Ratios are quite independent of the magnitude of the revolving line, and depend only on the absolute inclination of the two lines containing the angle to each other; and so for a given value of the angle are determinate.

The converse of this proposition (viz. that for a given value of the Trigonometrical Ratio, the angle is determinate) is only true with certain limitations, as will appear hereafter.

Obs. The necessity of the proviso *disregarding signs* will appear on reading the next section.

11. Let the angle AOP (fig. Art. 7) contain α degrees, then P_1OB, and OP_1N (since they either of them with AOP_1 make a right angle) contain $90-\alpha$ degrees each. Drawing P_1N' perpendicular to OB, we have by our definitions

$$\sin P_1OB = \frac{P_1N'}{OP_1} = \frac{ON}{OP_1} = \cos AOP_1,$$

$$\text{or } \sin(90^\circ - \alpha) = \cos\alpha.$$

Again, $$\tan P_1OB = \frac{P_1N'}{ON'} = \frac{ON}{P_1N} = \cot AOP_1,$$

$$\text{or } \tan(90^\circ - \alpha) = \cot\alpha,$$

$$\text{so also } \sec(90^\circ - \alpha) = \operatorname{cosec}\alpha,$$

and these equations may be proved to exist for any position of the revolving line OP.

It is from these relations, viz. that the cosine, cotangent, cosecant of an angle, are respectively the same as the sine, tangent, and secant of the complement, that they derive their names.

From the same relations we also have

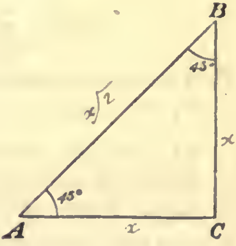
$$\sin a = \cos (90° - a),$$
$$\tan a = \cot (90° - a),$$
$$\sec a = \operatorname{cosec} (90° - a).$$

Hence we can always express a Trigonometrical Ratio of the complement of an angle in terms of some Trigonometrical Ratio of the simple angle.

12. *The Trigonometrical Ratios of* 30°, 45°, 60°. The calculation of the values of the Trigonometrical Ratios of all angles is a matter of considerable labour, involving processes not described in this treatise, but they may be easily found in some particular cases.

Let ABC be an isosceles triangle having a right angle at C; then CAB = half a right angle = 45°.

Let $AC = x = BC$,
then $AB^2 = AC^2 + BC^2 = 2x^2$
or $AB = x\sqrt{2}$,



and we have

$$\sin 45° = \frac{BC}{AB} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\cos 45° = \frac{AC}{AB} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\tan 45° = \frac{BC}{AC} = \frac{x}{x} = 1,$$

$$\cot 45° = \frac{AC}{BC} = \frac{x}{x} = 1,$$

$$\sec 45° = \frac{AB}{AC} = \frac{x\sqrt{2}}{x} = \sqrt{2},$$

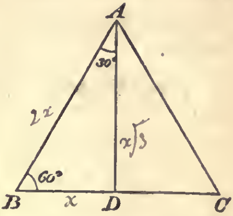
$$\operatorname{cosec} 45° = \frac{AB}{BC} = \frac{x\sqrt{2}}{x} = \sqrt{2}.$$

Again, let ABC be an equilateral triangle, then angle $ABC = \frac{180^\circ}{3} = 60^\circ$. Let AD be perpendicular to BC, then angle $BAD = 30^\circ$. Let $BD = x$, $AB = 2x$; then

$$AB^2 = AD^2 + BD^2,$$

$$\text{or } 4x^2 = AD^2 + x^2,$$

and therefore $AD = x\sqrt{3}$; and we have,



$$\sin 60^\circ = \cos 30^\circ = \frac{AD}{AB} = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \sin 30^\circ = \frac{BD}{AB} = \frac{x}{2x} = \frac{1}{2},$$

$$\tan 60^\circ = \cot 30^\circ = \frac{AD}{BD} = \frac{x\sqrt{3}}{x} = \sqrt{3},$$

$$\cot 60^\circ = \tan 30^\circ = \frac{BD}{AD} = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}},$$

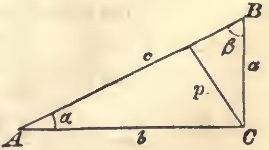
$$\sec 60^\circ = \operatorname{cosec} 30^\circ = \frac{AB}{BD} = \frac{2x}{x} = 2,$$

$$\operatorname{cosec} 60^\circ = \sec 30^\circ = \frac{AB}{AD} = \frac{2x}{x\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

Thus we have found the values of all the Trigonometrical Ratios of 30°, 45°, and 60°.

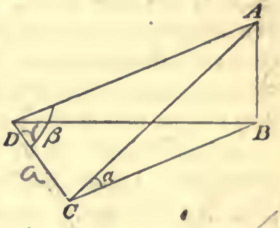
13. We will now assume that we know, and have arranged in Tables the values of the Trigonometrical Ratios of all angles, as well as of 30°, 45°, and 60°, and shew how they may be used to solve many interesting problems.

Let ABC be a right-angled triangle, C the right angle, and let angles CAB, CBA contain α°, β° respectively. Let the lengths of the sides opposite to A, B, C be a, b, c respectively. Then $\sin\alpha = \frac{a}{c}$, or $a = c \sin\alpha$; $\tan\alpha = \frac{a}{b}$, or $a = b \tan\alpha$;



$\cos \alpha = \frac{b}{c}$, or $b = c \cos \alpha$; also if p be the perpendicular from the angle C on the hypothenuse AB, we have $\frac{p}{b} = \sin \alpha = \frac{a}{c}$, or $cp = ab$; and similarly other relations may be found to exist between the sides and angles of a right-angled triangle. The student will best learn by practice to apply these relations to solve problems in each particular case; the following may be taken as an example.

A person on one side of a river at C wishes to determine the height of a tower AB on the other side. He observes, by an instrument made for the purpose, the angle which CA makes with the horizontal line CB. (Let this be α.) He then places a mark at C, and measures the distance CD (a feet) at right angles to CB, and when at D he observes the angle ADC (β suppose). He then calculates the height of the tower thus; ACD is a right-angled triangle, and therefore



$$AC = CD \tan ADC \quad \text{or} \quad AC = a \tan \beta.$$

So also ABC is a right-angled triangle, and therefore

$$AB = AC \sin ACB = AC \sin \alpha.$$

But
$$AC = a \tan \beta,$$
$$\therefore \; AB = a \tan \beta \sin \alpha.$$

Hence since we know the values of a, $\tan \beta$, $\sin \alpha$, we can calculate the height of the tower.

Ex. $a = 50$ yards, $\beta = 45°$, $\alpha = 30°$,

height of tower $= 50 \times 1 \times \frac{1}{2} = 25$ yards.

Instead of observing the $\angle ADC$, we might have observed the $\angle BDC$ (γ suppose), and proceeded as follows,

$$BC = DC \tan BDC = a \tan \gamma,$$
$$AB = BC \tan ACB = a \tan \gamma \tan \alpha.$$

This method would also give the breadth of the river, $BC = a \tan \gamma$.

14. The two principal instruments by which observations are taken in surveying are the sextant and the theodolite. The construction of these instruments depends on certain properties of light, the discussion of which falls under the science of optics. By the sextant we are enabled to observe the angle which any two objects subtend at the eye of the observer, as for instance the angle ADC in the case above.

By the theodolite we can observe the elevation of any object, that is to say, the angle which the line joining the object and the eye of the observer makes with the horizontal plane, as the angle ACB in the case above. We can also observe the angle which two objects in the same horizontal plane subtend at the eye of the observer; the sextant might do this, if we knew that the objects were in the same horizontal plane; the theodolite tells us that they are, as well as the amount of the angle they subtend. When an object is lower than the observer, the angle which the line joining it and the eye of the observer makes with the horizontal plane is called its depression.

EXAMPLES. (A).

1. Find the number of degrees in the angle $17^{g}, 34^{\backprime}, 27^{\backprime\backprime}$.

We have $17^{g}, 34^{\backprime}, 27^{\backprime\backprime} = 17^{g}.3427,$

and $\dfrac{D}{90} = \dfrac{G}{100}$, or $D = \dfrac{9}{10} \times 17.3427 = 15.60843$ degrees

$= 15^{\circ}, 36', 30''.348.$ *Ans.* $15^{\circ}, 36', 30''.348.$

2. If 7 feet is the unit of length, by what number is 1421 yards represented? *Ans.* 609.

3. If 3 inches is the unit of length, what is the unit of superficies? *Ans.* 9 square inches.

4. If 1728 square inches is represented by the number 108, what is the unit of length? *Ans.* 4 inches.

5. Reduce to grades, &c. the following angles, 30°; $22^\circ, 30'$; 18°; $29^\circ, 30'$; $37^\circ, 22', 19''$.

Answers. $33^g, 33^{`}, 33^{``}.\dot{3}$; $25^g$; $20^g$; $32^g, 77^{`}, 77^{``}.\dot{7}$; $41^g, 52^{`}, 43^{``}\frac{67}{81}$.

6. Reduce to degrees, &c. the angles, $27^g, 5^{`}, 22^{``}$; $46^g, 17^{`}, 19^{``}$.
Answers. $24^\circ, 20', 49''.128$; $41^\circ, 33', 16''.956$.

7. What is the unit of measurement when $66\frac{2}{3}$ grades is represented by 20? *Ans.* 3 degrees.

8. If $\frac{1}{8}$th of a right angle be the unit, what is the value of an angle 5.09296 in degrees? *Ans.* 57.2958.

9. Find all the angles less than a right angle, which can be expressed by an integral number both of degrees and grades.

10. Find the complements and supplements of the following angles, 27°; $29^\circ, 13', 15''$; $47^\circ, 29', 47''$; $107^\circ, 22'$; $84^\circ, 10'$.

11. One angle of a right-angled triangle is 10° greater than the other; find them. *Ans.* $40^\circ, 50^\circ$.

12. One angle of a right-angled triangle is three times as great as the other; find them. *Ans.* $67^\circ, 30'$; $22^\circ, 30'$.

13. The three angles of a triangle are in A.P. and the common difference is 12°; find them. *Ans.* $48^\circ, 60^\circ, 72^\circ$.

14. Of the three angles of a triangle, one is double either of the other two; find them. *Ans.* $90^\circ, 45^\circ, 45^\circ$.

15. The three angles of a triangle are in the ratio 2 : 3 : 5; find them. *Ans.* $36^\circ, 54^\circ, 90^\circ$.

16. In a triangle, twice the difference between the first and third angle is equal to the sum of the second and third; and the third is equal to three times the difference between the first and second; find them. *Ans.* 80, 70, 30.

17. In the isosceles triangle of Euclid IV. 10, find the angles. *Ans.* $36^\circ, 72^\circ, 72^\circ$.

18. The vertical angle of an isosceles triangle being 80°, find the other angles. *Ans.* $50''$.

19. The sides of a right-angled triangle are in A.P., shew that they are in the ratio 3 : 4 : 5.

20. If one angle of a right-angled triangle is 39°, and the hypothenuse is 7 feet, find the other sides.

Ans. 4.4052428, 5.440022 feet.

21. If one angle of a right-angled triangle is 26°, and the side adjacent to it 5 feet, find the other sides.

Ans. 5.5630095, 2.438663 feet.

22. If one angle of a right-angled triangle is 37°, 10′, and the side opposite to it 124 feet, find the other sides.

Ans. 205.25193, 163.5614684 feet.

23. If the two sides of a right-angled triangle are 30 feet and 40 feet, find the hypothenuse and the angles.

Ans. 50 feet; 36°, 53′; 53°, 7′.

24. If the hypothenuse and one side of a right-angled triangle are 13 feet and 5 feet, find the other side and the angles.

Ans. 12 feet; 67°, 23′; 22°, 37′.

25. If two sides of a right-angled triangle are 6 feet and 8 feet, find the perpendicular on the hypothenuse from the right angle. *Ans.* 4.8 feet.

26. If one angle of a right-angled triangle is 25°, and the hypothenuse 7 feet, find the perpendicular on the hypothenuse.

Ans. 2.6811554.

27. A ladder is set 4 feet from a wall, and just reaches a window 20 feet from the ground, find its length.

Ans. 20.39... feet.

28. The length of an upright stick is 6 feet, and the length of its shadow 12 feet, find the altitude, or angle of elevation, of the sun. *Ans.* 26°, 34′.

29. The length of a kite string is 300 yards, and the elevation of the kite is 32°, find its height. *Ans.* 158.97579 yards.

30. A person in a balloon whose elevation is 37° drops a stone, which falls 560 yards from the observer, find the height of the balloon. *Ans.* 421.990296 yards.

31. Construct the angles whose tangents are 3, and $\frac{2}{3}$ respectively.

32. Construct the angles whose sines are $\frac{1}{3}$ and $\frac{3}{5}$ respectively.

33. A church tower on the side of a river is 200 feet high, and its elevation from the other side is 29°, find the breadth of the river. *Ans.* 360.80956 feet.

34. A hill whose slope makes an angle of 12° with the horizon is one mile long, find the height of the hill.

Ans. 1097.773776 feet.

35. If the road up the hill in the last question were made obliquely so as to be two miles long instead of one, find the inclination of the road to the horizon. *Ans.* 5°, 58′.

36. The elevation of a tower is 30° to a man 6 feet high, 140 feet from the foot of the tower, find its height.

Ans. 86.8 feet nearly.

EXAMPLES. (B).

1. Construct the angle whose sine is $\frac{1}{\sqrt{3}}$.

Take BD (fig. Art. 12) $= x$, then $AD = x\sqrt{3}$; with centre B, distance equal to AD, describe a circle meeting AD in E, join BE, then

$$\sin BED = \frac{BD}{BE} = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}},$$

and the angle BED has been found as required.

2. Find the linear unit, when an acre is 100,000 square units. *Ans.* 1 link.

3. If a rectangle 3 yards long by 2 broad be taken as the unit of superficies, what quantity will represent 2 square feet?

Ans. $\frac{1}{27}$.

4. How is 37 yards, 2 feet, 8 inches represented, when the unit of length is 4 inches? *Ans.* By 341.

5. If the angle of an equilateral triangle be taken as the unit of measurement, how many degrees will be contained in the angle $.\dot{3}$?

6. Find the number of grades in $27^\circ, 35', 24''$.

Ans. $30^g.6\dot{5}$.

7. What is the supplement of $227^g, 35^`, 46^{``}$?

8. The complement of $45^\circ \pm \alpha$ is $45^\circ \mp \alpha$.

9. The difference of two acute angles of a right-angled triangle is δ°, find the angles.

10. The vertical angle of an isosceles triangle is α°, find the angles at the base.

11. The length of a person's shadow is twice his height, find the altitude of the sun, having given $\tan 26^\circ, 34' = .5$.

12. Shew that $(\sin 60^\circ - \sin 45^\circ)(\cos 30^\circ + \cos 45^\circ) = \sin^2 30^\circ$.

13. Shew that $(\sin 30^\circ + \cos 30^\circ)(\sin 60^\circ - \cos 60^\circ) = \cos 60^\circ$.

14. Shew that $\dfrac{\sin 45^\circ - \sin 30^\circ}{\sin 45^\circ + \sin 30^\circ} = (\sec 45^\circ - \tan 45^\circ)^2$.

15. Determine the height of an object whose elevation is 40°, the observer being 140 feet distant, and his eye 5 feet from the ground, having given that $\tan 40^\circ = .85$.

Ans. 124 feet.

16. Construct the angle whose tangent is $\sqrt{2}$.

17. Construct the angle whose sine is $\dfrac{\sqrt{2}-1}{2}$.

18. A ladder of length l is placed against a wall so that the angle it makes with the ground is double that which it makes with the wall: find how far from the wall the foot of the ladder is.

Ans. $\dfrac{l}{2}$.

19. The town C is halfway between the towns D and E; and the towns C, E, F are equidistant from each other. If distance from D to E is 12 miles, find the distance from D to F.

Ans. $6\sqrt{3}$ miles.

20. A ladder 20 feet long just reaches the top of a wall when its foot is 13 feet from the foot of the wall; shew that when its foot is 5 feet from the wall, the ladder projects 4 feet beyond the top of the wall.

21. In a right-angled triangle the lengths of lines drawn from the acute angles to the middle points of the opposite sides are d, d' respectively. Find the sides of the triangle.

Ans. $\frac{2}{\sqrt{15}}\sqrt{(4d^2-d'^2)}$, $\frac{2}{\sqrt{15}}\sqrt{(4d'^2-d^2)}$.

22. A person travelling southwards observes two objects towards the S.E. After 8 miles travelling, one of them is N.E., and the other E. Their distances from him are then $4\sqrt{2}$ miles, and 8 miles respectively.

23. Whilst sailing due West, I observe two ships at anchor directly North of me: after sailing 6 miles the directions of the ships make angles $60°$, $30°$ with my course respectively. The distance between them is $4\sqrt{3}$ miles.

24. A staff 1 foot long stands on the top of a tower 200 feet high. Shew that the angle it subtends at a place 100 feet from the foot of the tower is $6'$.

Given $\tan 63°.27' = 2$, $\tan 63°.33' = 2.01$.

25. The angles of depression of the top and bottom of a column observed from a tower 108 feet high are $30°$, $60°$ respectively. Shew that the height of the column is 72 feet.

26. From the top of a column 100 feet high, the angles of depression of two objects in a line with the column, and in the horizontal plane on which the column stands, are observed to be $30°$ and $60°$. The distance between them is $\frac{200}{\sqrt{3}}$ feet.

27. The length of a road, in which the ascent is 1 foot in 5, from the foot of a hill to the top is a mile and two-thirds: what will be the length of a zigzag road in which the ascent is 1 foot in 12? *Ans.* 4 miles.

EXAMPLES. (C).

1. The same line is represented by m and n in two systems of measurement, compare the units of length in the two systems.

2. What is the greatest unit of length with which .625 feet may be represented by an integer?

3. Find the number of French minutes in one English minute, and the number of French seconds in one English second.

Ans. $1^{`}.\dot{8}5\dot{1}$, $3^{``}.\dot{0}8641975\dot{3}$.

4. The angles of a triangle are in A. P.; shew that one of them must equal 60°.

5. The supplement of one angle of a triangle is double the complement of another, and triple that of the third: find the angles. *Ans.* $81\frac{9}{11}$, $40\frac{10}{11}$, $57\frac{3}{11}$ degrees.

6. If with two units of angular measurement differing by 10^g the measures of an angle are as $3:2$, what are those units?

Ans. 20^g, 30^g.

7. If the measure of an angle equal the sum of the number of degrees and half the grades in it, what is the unit of angular measure? *Ans.* $\left(\frac{9}{14}\right)^\circ$.

8. Three angles are in A. P. The number of grades in the greatest is equal to the number of degrees in the sum of the other two: shew that the angles are in the ratio $11:19:27$.

9. In an isosceles triangle whose sides are a, a, c, if p be the perpendicular from one of the angles at the base on the opposite side, then $p=\frac{c\sqrt{4a^2-c^2}}{2a}$.

10. A ladder (whose inclination to the ground is α) just reaches the top of a wall; at a distance a feet up the ladder the wall subtends an angle 2α; shew that the height of the wall is $2a \sin \alpha$.

11. A person observes the angular elevation of a column; after approaching a feet, he finds its elevation doubled; again approaching b feet it is again doubled. Shew that the second point of observation is $\frac{a^2}{2b}$ feet from the foot of the tower.

If $a = b\sqrt{3}$, shew that the first elevation was 15°.

12. Construct the angle whose tangent is $\sqrt{5} - 1$.

13. At the foot of a mountain the elevation of its summit is 45°. After ascending 1 mile up a slope of 30° its elevation is found to be 60°. The height of the mountain is $\frac{\sqrt{3}+1}{2}$ miles.

14. A person at the edge of a river observes the elevation α of a tower on the other side; on retreating a feet he finds its elevation to be β: the height of the tower is $\frac{a \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}$, the breadth of the river is $\frac{a \tan \beta}{\tan \alpha - \tan \beta}$.

15. A rock is observed from a ship to bear N. N. W.; after sailing 10 miles in direction E.N.E. the rock is due West: its distance from the ship at the first observation was $10(\sqrt{2}-1)$ miles. Given $\tan 22\frac{1}{2}° = \sqrt{2} - 1$.

16. A ship sailing North sees two lighthouses due East. After sailing an hour they are S.E., and S.S.E.; the distance between them is 8 miles: find the rate of the ship.

Ans. 13.6 miles nearly.

17. Two persons A and B start from two points distant 400 yards. B starts at right angles to the line joining the two points at the rate of 90 yards a minute. A starts in a direction to catch B as soon as possible at the rate of 150 yards a minute; find how

long he will be before he catches him, and the direction in which he will walk, having given $\sin 36°.\ 53' = \frac{3}{5}$.

Ans. 3 minutes, 20 seconds.

18. A tower is situated at the top of a hill whose inclination is 30°. The angle subtended by the tower at the foot of the hill is 15°; and on walking a yards up the hill it is found to be 30° The height of the tower is $\frac{a}{\sqrt{3}}$ feet.

19. From two points in the diameter of a circle produced tangents are drawn to the circle. Given the distance c between the points, and the inclinations α, β of the tangents to the diameter, shew that the radius of the circle is $\frac{c}{\operatorname{cosec}\beta - \operatorname{cosec}\alpha}$.

20. A person travelling along a road observes the elevation α of a tower the nearest distance (a) of which from the road is known; at the same time he observes the angular distance β of the top of the tower from an object in the road. The height of the tower is $\frac{a \sin \alpha}{\sqrt{(\cos^2 \alpha - \cos^2 \beta)}}$.

21. At three positions in the same horizontal plane distant from each other 60, 80, 100 feet respectively, the elevation of a tower is observed to be 45°; find its height. *Ans.* 50 feet.

22. Two spectators at two stations, distant $2a$ from each other, observe the elevation of a kite to be α at each station, and the angle subtended by the kite and the other station to be β; shew that the height of the kite is $a \sec \beta \sin \alpha$.

23. A person standing on the top A of a light-house AB, of known height 300 feet, observes a ship sailing from C to D in a straight line: he knows that CD is perpendicular to the plane ACB, and he observes the angles $CAD = 30°$, $BAD = 60°$. Shew that $CD = 300$ feet; $BD = 300\sqrt{3}$ feet.

24. The altitude of the sun is 45°, and it is at the point of the compass 60° from the south. The breadth of the shadow of a south wall is one-half its height.

25. The angular altitude and breadth of a cylindrical tower are observed to be α and β respectively; but at a point a feet nearer the foot of the tower they are α', β' respectively: find the height and radius of the tower.

Ans. Height $= \dfrac{a}{\cot\alpha - \cot\alpha'}$, radius $= \dfrac{a}{\operatorname{cosec}\dfrac{\beta}{2} - \operatorname{cosec}\dfrac{\beta'}{2}}$.

26. A person a feet from the foot of a tower (height h) sees a mountain summit in a line with the top of the tower. At the foot of the tower the elevation of the mountain is α: shew that the height of the mountain is

$$\frac{ah}{a - h\cot\alpha}.$$

27. An embankment (the height of whose summit is c) stands on a plain, and a man looking directly up the slope of the embankment just sees the top of a tower on the plain: when at distance a from the summit the man's eye is level with the summit, and when at the distance b from the summit he can just see the foot of the tower. Shew that the height of the tower is $\dfrac{ac}{a-b}$.

SECTION II.

TRIGONOMETRICAL RATIOS OF ANGLES GREATER THAN 90°. RELATIONS AMONGST THE RATIOS. CHANGE IN SIGN AND MAGNITUDE OF THE RATIOS THROUGH THE FOUR QUADRANTS. CHANGE FROM POSITIVE TO NEGATIVE INFINITY. OTHER TRIGONOMETRICAL RATIOS.

15. We now proceed to express the Trigonometrical Ratios of all angles in terms of Trigonometrical Ratios of positive angles less than a right angle.

We will first find the Trigonometrical Ratios of the supplement of an angle.

Let $A'OP_2 = AOP_1$ absolutely (fig. Art. 7): then AOP_2 is the supplement of AOP_1 or a, that is, $AOP_2 = 180° - a$.

Let $ON = x, \quad P_1N = y, \quad OP_1 = r,$

then $OM = -x, \quad P_2M = y, \quad OP_2 = r.$

Hence
$$\sin(180° - a) = \frac{y}{r} = \sin a,$$

$$\cos(180° - a) = \frac{-x}{r} = -\cos a,$$

$$\tan(180° - a) = \frac{y}{-x} = -\tan a,$$

$$\cot(180° - a) = \frac{-x}{y} = -\cot a,$$

$$\sec(180° - a) = \frac{r}{-x} = -\sec a,$$

$$\operatorname{cosec}(180° - a) = \frac{r}{y} = \operatorname{cosec} a;$$

which equations give the Trigonometrical Ratios of the supplement of an angle, in terms of the Trigonometrical Ratios of the simple angle.

We may observe that they all change in sign except the sine and cosecant.

16. We will next find the Trigonometrical Ratios of a negative angle.

Let $AOP_4 = AOP_1$ absolutely: then $AOP_4 = -\alpha$;

also $ON = x, \quad P_4N = -y, \quad OP_4 = r.$

Hence
$$\sin(-\alpha) = \frac{-y}{r} = -\sin\alpha,$$
$$\cos(-\alpha) = \frac{x}{r} = \cos\alpha,$$
$$\tan(-\alpha) = \frac{-y}{x} = -\tan\alpha,$$
$$\cot(-\alpha) = \frac{x}{-y} = -\cot\alpha,$$
$$\sec(-\alpha) = \frac{r}{x} = \sec\alpha,$$
$$\operatorname{cosec}(-\alpha) = \frac{r}{-y} = -\operatorname{cosec}\alpha;$$

which equations give the Trigonometrical Ratios of a negative angle in terms of those of the positive angle of the same absolute magnitude.

We may observe that they all change in sign except the cosine and secant.

17. Let $A'OP_3 = AOP_1 = \alpha$, then $AOP_3 = 180° + \alpha$,

also $OM = -x, \quad P_3M = -y, \quad OP = r.$

Hence
$$\sin(180° + \alpha) = \frac{-y}{r} = -\sin\alpha,$$
$$\cos(180° + \alpha) = \frac{-x}{r} = -\cos\alpha,$$
$$\tan(180° + \alpha) = \frac{-y}{-x} = \tan\alpha,$$
$$\cot(180° + \alpha) = \frac{-x}{-y} = \cot\alpha,$$
$$\sec(180° + \alpha) = \frac{r}{-x} = -\sec\alpha,$$
$$\operatorname{cosec}(180° + \alpha) = \frac{r}{-y} = -\operatorname{cosec}\alpha.$$

Here all the ratios change in sign except the tangent and cotangent.

18. It is evident that, when the revolving line OP has reached OP_1 the second time, that is, has described the angle $(360^\circ + \alpha)$, all the Trigonometrical Ratios are the same as those of the angle α, for the triangle which determines their values is identical in the two cases. The same remark applies to any number of revolutions, so that we may say generally that the Trigonometrical Ratios of $(n\,.\,360^\circ + \alpha)$ are equal severally to those of α.

19. The formulæ in Articles 12, 15, 16, 17, 18, have been proved for the case in which α is less than 90°; that they are true for all values whatever of α the student may satisfy himself by drawing figures to represent the different cases: for a more logical proof that these forms are true for all angles whatever he must be content to wait till he is further advanced, as the reasoning is of too subtle a nature for the present.

These formulæ may also be classed as follows:

$$\sin\alpha = \sin(180^\circ - \alpha) = -\sin(180^\circ + \alpha) = -\sin(-\alpha) = \sin(n\,.\,360^\circ + \alpha),$$

$$\cos\alpha = -\cos(180^\circ - \alpha) = -\cos(180^\circ + \alpha) = \cos(-\alpha) = \cos(n\,.\,360^\circ + \alpha),$$

$$\tan\alpha = -\tan(180^\circ - \alpha) = \tan(180^\circ + \alpha) = -\tan(-\alpha) = \tan(n\,.\,360^\circ + \alpha),$$

$$\cot\alpha = -\cot(180^\circ - \alpha) = \cot(180^\circ + \alpha) = -\cot(-\alpha) = \cot(n\,.\,360^\circ + \alpha),$$

$$\sec\alpha = -\sec(180^\circ - \alpha) = -\sec(180^\circ + \alpha) = \sec(-\alpha) = \sec(n\,.\,360^\circ + \alpha),$$

$$\operatorname{cosec}\alpha = \operatorname{cosec}(180^\circ - \alpha) = -\operatorname{cosec}(180^\circ + \alpha) = -\operatorname{cosec}(-\alpha) = \operatorname{cosec}(n\,.\,360^\circ + \alpha).$$

We obtain also from the above formulæ the signs which the Trigonometrical Ratios of any angles must have according to the quadrant in which their bounding line is situated.

20. We are now in a position to express the Trigonometrical Ratios of any angle whatever in terms of the Ratios of a positive angle less than 90°.

If the angle is negative make it positive by formulæ in Art. 16.

If the angle is greater than 360°, suppress 360° as often as necessary to make it less than 360° by Art. 18.

If the angle is now greater than 180°, suppress 180° by Art. 17.

If the angle is now greater than 90° take its supplement by Art. 15.

When the angle is negative labour is sometimes saved by adding 360° once or oftener, till the angle is positive, instead of using Art. 16.

Ex. $\tan 995^\circ = \tan 275^\circ = \tan 95^\circ = -\tan 85^\circ$,

$\operatorname{cosec}(-995^\circ) = -\operatorname{cosec} 995^\circ = -\operatorname{cosec} 275^\circ = \operatorname{cosec} 95^\circ = \operatorname{cosec} 85^\circ$,

or thus, $\operatorname{cosec}(-995^\circ) = \operatorname{cosec}(1080^\circ - 995^\circ) = \operatorname{cosec} 85^\circ$.

21. From the formulæ in Arts. 15, 16, 17, 18 we deduce the following, which are true for all values of the angle,

$$\sin \alpha \operatorname{cosec} \alpha = 1,$$

$\therefore \text{Sin}\,\alpha = \frac{1}{\text{Cosec}\,\alpha}$ Cosec α =

$$\cos \alpha \sec \alpha = 1,$$

Sin α

$$\tan \alpha \cot \alpha = 1.$$

Also in the first quadrant $\tan \alpha = \dfrac{y}{x} = \dfrac{\frac{y}{r}}{\frac{x}{r}} = \dfrac{\sin \alpha}{\cos \alpha}$,

in the second quadrant $\tan \alpha = \dfrac{y}{-x} = \dfrac{\frac{y}{r}}{-\frac{x}{r}} = \dfrac{\sin \alpha}{\cos \alpha}$,

in the third quadrant $\tan \alpha = \dfrac{-y}{-x} = \dfrac{-\frac{y}{r}}{-\frac{x}{r}} = \dfrac{\sin \alpha}{\cos \alpha}$,

in the fourth quadrant $\tan \alpha = \dfrac{-y}{x} = \dfrac{-\frac{y}{r}}{\frac{x}{r}} = \dfrac{\sin \alpha}{\cos \alpha}$.

Hence always $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha};$$

and also $$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

Also since $$x^2 + y^2 = r^2, \qquad \text{(Euclid, I. 47)}$$

divide by r^2, and we have

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

or $$\cos^2\alpha + \sin^2\alpha = 1\ ;$$

whence $$\sin\alpha = \pm\sqrt{(1-\cos^2\alpha)};$$

$$\cos\alpha = \pm\sqrt{(1-\sin^2\alpha)}:$$

divide by x^2, and we have

$$\frac{r^2}{x^2} = 1 + \frac{y^2}{x^2},$$

or $$\sec^2\alpha = 1 + \tan^2\alpha\ ;$$

whence $$\sec\alpha = \pm\sqrt{(1+\tan^2\alpha)};$$

$$\tan\alpha = \pm\sqrt{(\sec^2\alpha - 1)}:$$

divide by y^2, and we have,

$$\frac{r^2}{y^2} = 1 + \frac{x^2}{y^2},$$

or $$\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha\ ;$$

whence $$\operatorname{cosec}\alpha = \pm\quad(1+\cot^2\alpha);$$

$$\cot\alpha = \pm\quad(\operatorname{cosec}^2\alpha - 1).$$

Also we have $$\sin\alpha = \frac{y}{r} = \pm\frac{y}{\sqrt{(x^2+y^2)}} = \pm\frac{\frac{y}{x}}{\sqrt{\left(1+\frac{y^2}{x^2}\right)}}$$

$$= \pm\frac{\tan\alpha}{\sqrt{(1+\tan^2\alpha)}};$$

so also $$\cos\alpha = \pm\frac{1}{\sqrt{(1+\tan^2\alpha)}};$$

$$\sin\alpha = \pm\frac{1}{\sqrt{(1+\cot^2\alpha)}};$$

$$\cos a = \pm \frac{\cot a}{(1+\cot^2 a)}.$$

The upper or lower sign must be used in these forms according to the quadrant in which the angle lies.

Obs. Some of these expressions are really included in the others, for they are derived from the same relation $x^2+y^2=r^2$. We may also put $90-a$ for a in any of these relations, and they must still remain true; thus the relation

$$\sin a = \pm \frac{\tan a}{\sqrt{(1+\tan^2 a)}}$$

becomes $$\sin(90-a) = \pm \frac{\tan(90-a)}{\sqrt{\{1+\tan^2(90-a)\}}};$$

or, $$\cos a = \pm \frac{\cot a}{\sqrt{(1+\cot^2 a)}},$$

as we obtained before. In fact, whatever *general* relation holds between any Trigonometrical Ratios, must necessarily hold if the complementary or supplementary ratios are substituted for them.

22. We stated above (Art. 10) that when we know the value of any of the Trigonometrical Ratios, the sine for instance, we only know the value of the corresponding angle within certain limitations: we can now more fully explain this. From the preceding Articles we may observe, that whenever the revolving line OP (fig. Art. 7) has come into either the position OP_1, or OP_2, the value of the sine of the angle described is the same and of the same sign. Hence, if the value and sign of the sine is given, and the sine of AOP_1 is equal to this given value and of the same sign, we know that the angle whose sine is given is bounded by one of the lines OP_1, OP_2; but of the number of revolutions the line OP has taken before reaching the position OP_1 or OP_2, and whether these revolutions have been in a positive or negative direction we are quite ignorant. Similar remarks will apply to all the Trigonometrical Ratios.

23. As an illustration of the principles laid down in the preceding Articles, we will trace the changes in sign and mag-

nitude of the Trigonometrical Ratios as the angle (fig. Art. 7) AOP or α passes through all values from 0° to 360°. Let

$$ON = x, \quad PN = y, \quad OP = r,$$

for all positions of OP: then r is always positive and never less than x and y, and therefore the sine and cosine are never greater than 1, and the secant and cosecant never less than 1: and since x and y may have any ratio to each other, the tangent and cotangent may have any values whatever.

When OP coincides with OA, $x = r$, $y = 0$.

Hence, $\sin 0° = \frac{0}{r} = 0$; $\tan 0° = \frac{0}{r} = 0$; $\sec 0° = \frac{r}{r} = 1$;

$$\cos 0° = \frac{r}{r} = 1\,; \ \cot 0° = \frac{r}{0} = \infty\,; \ \operatorname{cosec} 0° = \frac{r}{0} = \infty\,.$$

As OP revolves from OA to OB, x diminishes, y increases, and both are positive; hence the sine, tangent, secant increase, the cosine, cotangent, cosecant diminish, and they are all positive.

When OP coincides with OB, $x = 0$, $y = r$.

Hence, $\sin 90° = \frac{r}{r} = 1$; $\tan 90° = \frac{r}{0} = \infty$; $\sec 90° = \frac{r}{0} = \infty$;

$$\cos 90° = \frac{0}{r} = 0\,; \ \cot 90° = \frac{0}{r} = 0\,; \ \operatorname{cosec} 90° = \frac{r}{r} = 1.$$

As OP revolves from OB to OA', x increases and is negative, y diminishes and is positive; hence the sine, tangent, secant diminish, the cosine, cotangent, cosecant increase; also the sine and cosecant are positive, the rest negative.

When OP coincides with OA', $x = -r$, $y = 0$.

Hence,

$$\sin 180° = \frac{0}{r} = 0\,; \ \tan 180° = \frac{0}{-r} = 0\,; \ \sec 180° = \frac{r}{-r} = -1\,;$$

$$\cos 180° = \frac{-r}{r} = -1\,; \ \cot 180° = \frac{-r}{0} = -\infty\,; \ \operatorname{cosec} 180° = \frac{r}{0} = \infty\,.$$

As OP revolves from OA' to OB, x diminishes and is negative, y increases and is negative; hence the sine, tangent, secant increase, and the cosine, cotangent, cosecant diminish, also the tangent and cotangent are positive, the rest negative.

When OP coincides with OB', $x=-0$, $y=-r$.

Hence,

$$\sin 270^\circ = \frac{-r}{r} = -1\,;\ \tan 270^\circ = \frac{-r}{-0} = \infty\,;\ \sec 270^\circ = \frac{r}{-0} = -\infty\,;$$

$$\cos 270^\circ = \frac{-0}{r} = 0\,;\ \cot 270^\circ = \frac{-0}{-r} = 0\,;\ \operatorname{cosec} 270^\circ = \frac{r}{-r} = -1.$$

As OP revolves from OB' to OA, x increases and is positive, and y diminishes and is negative. Hence the sine, tangent, secant diminish, and the cosine, cotangent, cosecant increase; also the cosine and secant are positive, the rest negative.

When OP coincides with OA the second time, the values of the Trigonometrical Ratios become equal to those of 0°.

24. We have used above the relation $x=-0$, and so made

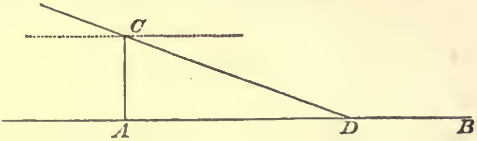
$$\sec 270^\circ = \frac{r}{x} = \frac{r}{-0} = -\infty\,,$$

instead of $+\infty$ which it would have been if we had put $x=0$. The reason of this is, that x has been continually diminishing and negative, and therefore the secant has been a continually increasing negative quantity; and we have thought it better to say that as the angle reaches 270°, the secant becomes an infinitely great negative quantity. If however the line OP had approached the position OB' from OA, by revolving in the negative direction, the secant of the angle -90° would be $+\infty$.

Thus for the same position of OP, supposed to revolve in opposite directions to reach this position, we have two values of the secant of the angle of opposite signs. In fact, positive and negative infinity seem to have a near approach to each other, for we find the Ratios which become infinite always pass from one to the other. The case above and similar ones are only in accordance with our preceding formulæ; for since 270° and -90° are supplementary angles, it follows that their secants should be equal and of opposite signs.

The following illustration may help to remove the difficulty from the Student's mind.

Let CA be drawn perpendicular to AB a line of unlimited length, CD drawn through C to meet AB in D. Let $AD = x$.



Now if angle ACD is made gradually to increase, the value of x increases without limit and is positive till ACD becomes a right angle. As soon as ACD becomes greater than a right angle, the value of x becomes suddenly an infinitely large negative quantity, and then diminishes to zero. The change in position of the point D for a small change in the revolving line, becomes infinitely great at this position of the line, though in general it is but gradual.

25. Besides the Trigonometrical Ratios above enumerated, the following are sometimes used.

The versine of the angle, which equals 1 – cosine, that is,

$$\text{vers}\, a = 1 - \cos a.$$

The coversine, which is the versine of the complement, that is,

$$\text{covers}\, a = 1 - \cos(90 - a) = 1 - \sin a.$$

The suversine, which is the versine of the supplement, that is,

$$\text{suvers}\, a = 1 - \cos(180 - a) = 1 + \cos a.$$

Also the chord of the angle, which is twice the sine of half the angle, that is,

$$\text{chd}\, a = 2 \sin \frac{a}{2}.$$

If we turn to fig. Art. 7, we see that

$$\text{vers}\, a = \frac{AN}{AO}, \quad \text{covers}\, a = \frac{BN'}{AO}, \quad \text{suvers}\, a = \frac{A'N}{AO};$$

and if AP_1 be joined, chord $a = \dfrac{AP_1}{AO}$.

26. The following notation is also used: if the sine of the angle α is a, that is, if $\sin\alpha = a$, then the angle whose sine is a is written $\sin^{-1}a$, that is, $\alpha = \sin^{-1}a$; and so for the other Trigonometrical Ratios.

Also, since $\alpha = \sin^{-1}a$; $\therefore$ $\sin\alpha = a$, $\cos\alpha = \sqrt{1-a^2}$,

$\tan\alpha = \dfrac{a}{\sqrt{1-a^2}}$. Hence $\alpha = \sin^{-1}a = \cos^{-1}\sqrt{1-a^2} = \tan^{-1}\dfrac{a}{\sqrt{1-a^2}}$.

EXAMPLES. (A).

1. Express in the corresponding Trigonometrical Ratios of angles less than 90°, $\sin 732°$, $\sin 2451°$, $\cos 674°$, $\cos(-424°)$, $\tan 582°$, $\tan(-1975°)$, $\cot 873°$, $\operatorname{cosec} 565°$, $\sec(-1892°)$.

Ans. $\sin 12°$, $-\sin 69°$, $\cos 46°$, $\cos 64°$, $\tan 42°$, $\tan 5°$, $-\cot 27°$, $-\operatorname{cosec} 25°$, $-\sec 88°$.

2. If $\sin\alpha = \dfrac{3}{5}$, find $\cos\alpha$, $\sec\alpha$, $\tan\alpha$.

Ans. $\cos\alpha = \dfrac{4}{5}$, $\sec\alpha = 1\dfrac{1}{4}$, $\tan\alpha = \dfrac{3}{4}$.

3. If $\cos\alpha = \dfrac{5}{13}$, find $\sin\alpha$, $\operatorname{cosec}\alpha$, $\cot\alpha$.

Ans. $\sin\alpha = \dfrac{12}{13}$, $\operatorname{cosec}\alpha = 1\dfrac{1}{12}$, $\cot\alpha = \dfrac{5}{12}$.

4. If $\sec\alpha = 3\dfrac{4}{7}$, find $\tan\alpha$, $\sin\alpha$.

Ans. $\tan\alpha = 3\dfrac{3}{7}$, $\sin\alpha = \dfrac{24}{25}$.

5. If $\tan\alpha = 2\dfrac{2}{5}$, find $\sec\alpha$, $\sin\alpha$.

Ans. $\sec\alpha = 2\dfrac{3}{5}$, $\sin\alpha = \dfrac{12}{13}$.

6. If $\sin\alpha = \dfrac{1}{3}$, find the other Ratios.

Ans. $\cos\alpha = \dfrac{2}{3}\sqrt{2}$, $\tan\alpha = \dfrac{1}{4}\sqrt{2}$, $\cot\alpha = 2\sqrt{2}$, $\sec\alpha = \dfrac{3}{4}\sqrt{2}$, $\operatorname{cosec}\alpha = 3$.

7. If $\tan a = \frac{2}{3}$, find $\sec a$, $\sin a$, vers a.

Ans. $\sec a = \frac{\sqrt{13}}{3}$, $\sin a = \frac{2}{\sqrt{13}}$, $\text{vers}\, a = \frac{\sqrt{13}-3}{\sqrt{13}}$.

8. In questions 4, 5, 6, 7 find the values of a from the Tables.

Ans. 73°, 45′; 67°, 23′; 19°, 28′; 33°, 41′.

Prove the truth of the following equations.

9. $\sin a \,\text{covers}\, a\,(1 + \text{cosec}\, a) = \cos^2 a$.

Here

$$\sin a \,\text{covers}\, a\,(1 + \text{cosec}\, a)$$

$$= \sin a\,(1 - \sin a)\left(\frac{1 + \sin a}{\sin a}\right) = 1 - \sin^2 a$$

$$= \cos^2 a. \quad \text{Q.E.D.}$$

10. $\cos a \tan a + \sin a \cot a = \sin a + \cos a$.

11. $\cos a \,\text{cosec}\, a = \cot a$.

12. $(1 + \tan^2 a)\cos^2 a = 1$.

13. $\frac{\sin a \sec a}{\tan a} = 1$.

14. $(\cos a \tan a)^2 + (\sin a \cot a)^2 = 1$.

15. $\frac{\cos a}{\sin a \cot^2 a} = \tan a$.

16. $\frac{\sec^2 a - 1}{\sec^2 a} = \sin^2 a$.

17. $\frac{\text{cosec}^2 a - 1}{\text{cosec}^2 a} = \cos^2 a$.

18. $(\sec^2 a - 1)(\text{cosec}^2 a - 1) = 1$.

19. $\sec a \,\text{cosec}\, a\,(\cos^2 a - \sin^2 a) = \cot a - \tan a$.

20. $\frac{\sin a \sec a}{\cos a \,\text{cosec}\, a} = \frac{\tan a}{\cot a}$.

21. $\frac{\sin a \cos a}{\sec a \,\text{cosec}\, a} = \sin^2 a - \sin^4 a$.

22. $\sec^2 a + \text{cosec}^2 a = \sec^2 a \,\text{cosec}^2 a$.

23. $\cot^2\alpha - \cos^2\alpha = \cos^2\alpha \cot^2\alpha$.

24. $\cos^4\alpha - \sin^4\alpha = \cos^2\alpha - \sin^2\alpha$.

25. $\tan\alpha + \cot\alpha = \sec\alpha \operatorname{cosec}\alpha$.

26. $\cos\alpha \operatorname{vers}\alpha\,(1 + \sec\alpha) = \sin^2\alpha$.

27. $\sin^2\alpha\,(1 + n\cot^2\alpha) = \cos^2\alpha\,(n + \tan^2\alpha)$.

28. $(\sin\alpha + \cos\alpha)(\tan\alpha + \cot\alpha) = \sec\alpha + \operatorname{cosec}\alpha$.

29. $\tan(-\alpha)\sec(-\alpha) = \dfrac{\tan\alpha}{\operatorname{vers}\alpha - 1}$.

30. If $\tan(\alpha + \beta) = \tan(\theta + \phi)$, then $\tan(\alpha - \theta) = \tan(\phi - \beta)$.

EXAMPLES. (B).

1. If $\tan\alpha = \dfrac{b}{a}$, then $\sin\alpha = \dfrac{b}{\sqrt{(a^2 + b^2)}}$, $\cos\alpha = \dfrac{a}{\sqrt{(a^2 + b^2)}}$,

for
$$\sin\alpha = \frac{\tan\alpha}{\sqrt{(1 + \tan^2\alpha)}} = \frac{\dfrac{b}{a}}{\sqrt{\left(1 + \dfrac{b^2}{a^2}\right)}} = \frac{b}{\sqrt{(a^2 + b^2)}},$$

$$\cos\alpha = \frac{1}{\sqrt{(1 + \tan^2\alpha)}} = \frac{1}{\sqrt{\left(1 + \dfrac{b^2}{a^2}\right)}} = \frac{a}{\sqrt{(a^2 + b^2)}}.$$

2. If $\tan\theta = \dfrac{1 + \tan\alpha}{1 - \tan\alpha}$, find $\sin\theta$, $\cos\theta$,

$$\sin\theta = \frac{1 + \tan\alpha}{\sqrt{\{2(1 + \tan^2\alpha)\}}} = \frac{1 + \tan\alpha}{\sec\alpha\,.\sqrt{2}} = \frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha),$$

$$\cos\theta = \frac{1 - \tan\alpha}{\sqrt{\{2(1 + \tan^2\alpha)\}}} = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha).$$

3. If $\sin\alpha = .\dot{3}$, shew that $\tan\alpha = \dfrac{1}{4}\sqrt{2}$, $\cot\alpha = 2\sqrt{2}$,

$$\sec\alpha = \frac{3}{4}\sqrt{2}.$$

4. If $\tan\alpha = .\dot{3}$, find $\sin\alpha$, $\cos\alpha$, $\operatorname{cosec}\alpha$, $\operatorname{vers}\alpha$.

Ans. $\frac{1}{10}\sqrt{(10)}$, $\frac{3}{10}\sqrt{(10)}$, $\sqrt{(10)}$, $\frac{\sqrt{(10)}-3}{\sqrt{(10)}}$.

Prove the truth of the following equations:

5. $\frac{\sin\alpha}{\tan\alpha}\cdot\frac{\cot\alpha}{\cos\alpha}=\frac{\operatorname{cosec}\alpha}{\sec\alpha}$.

6. $(\operatorname{cosec}\alpha-\cot\alpha)^2=\frac{1-\cos\alpha}{1+\cos\alpha}$.

7. $\cot\alpha+\frac{\sin\alpha}{1+\cos\alpha}=\operatorname{cosec}\alpha$.

8. $\sec\alpha\left\{1+\operatorname{cosec}\alpha\left(\cos^2\alpha-\sin\alpha\right)\right\}=\cot\alpha$.

9. $\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta=\sin^2\alpha-\sin^2\beta$.

10. $\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta=\cos^2\alpha-\sin^2\beta$.

11. $\cot\alpha-\sec\alpha\operatorname{cosec}\alpha\,(1-2\sin^2\alpha)=\tan\alpha$.

12. $\cot^2\alpha-\tan^2\alpha=(\cos^2\alpha-\sin^2\alpha)\sec^2\alpha\operatorname{cosec}^2\alpha$.

13. $\frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\tan\alpha\tan\beta$.

14. $\tan^2\alpha\tan^2\beta-1=\frac{\sin^2\alpha-\cos^2\beta}{\cos^2\alpha\cos^2\beta}$.

15. $\frac{1-\tan^2\alpha\tan^2\beta}{\tan^2\alpha\tan^2\beta}=\frac{\cos^2\alpha-\sin^2\beta}{\sin^2\alpha\sin^2\beta}$.

16. $\sin^2\alpha\tan^2\alpha+\cos^2\alpha\cot^2\alpha=\tan^2\alpha+\cot^2\alpha-1$.

17. $\sin^2\alpha\tan\alpha+\cos^2\alpha\cot\alpha+2\sin\alpha\cos\alpha=\sec\alpha\operatorname{cosec}\alpha$.

18. $\sec^4\alpha+\tan^4\alpha=1+2\sec^2\alpha\tan^2\alpha$.

19. $\operatorname{cosec}\alpha\,(\sec\alpha-1)-\cot\alpha\,(1-\cos\alpha)=\tan\alpha-\sin\alpha$.

20. Solve the equation $\sin\theta\cos\theta=\frac{2}{5}$.

$$\sin^2\theta\cos^2\theta=\frac{4}{25};$$

$$\therefore\ \sin^2\theta-\sin^4\theta=\frac{4}{25};$$

$$\therefore \sin^4\theta - \sin^2\theta + \frac{1}{4} = \frac{1}{4} - \frac{4}{25} = \frac{9}{100};$$

$$\therefore \sin^2\theta = \frac{1}{2} \pm \frac{3}{10} = \frac{5 \pm 3}{10} = \frac{1}{5} \text{ or } \frac{4}{5};$$

$$\therefore \sin\theta = \pm\frac{1}{\sqrt{5}} \text{ or } \pm\frac{2}{\sqrt{5}};$$

$$\therefore \theta = \sin^{-1}\left(\pm\frac{1}{\sqrt{5}}\right) \text{ or } \sin^{-1}\left(\pm\frac{2}{\sqrt{5}}\right).$$

Solve the following equations:

21. $2\sin\theta = \tan\theta$. *Ans.* $\theta = 0$, or 60°.

22. $2\sin^2\theta + 4\cos^2\theta = 3$. *Ans.* $\theta = \pm 45^\circ$.

23. $\tan\theta + 3\cot\theta = 4$. *Ans.* $\theta = 45^\circ$, or $\tan^{-1} 3$.

24. $8\sin\theta = 3\cos^2\theta$. *Ans.* $\theta = \sin^{-1}\frac{1}{3}$.

25. $3\sin\theta = 2\cos^2\theta$. *Ans.* $\theta = 30^\circ$.

26. $6\cot^2\theta - 4\cos^2\theta = 1$. *Ans.* $\theta = \pm 60^\circ$.

27. $\sin\theta + \operatorname{cosec}\theta = 2$. *Ans.* $\theta = 90^\circ$.

28. $\sec^2\theta - \frac{5}{2}\sec\theta + 1 = 0$. *Ans.* $\theta = 60^\circ$.

29. $\sec\theta = 2\tan\theta$. *Ans.* $\theta = 90^\circ$ or 30°.

30. Find the values of $\cos 585^\circ$, $\tan 600^\circ$, $\operatorname{cosec} 690^\circ$.

31. Shew that $\sin^{-1}\frac{4}{5} = \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3} = \sec^{-1}\frac{5}{3}$.

32. If $\sin(x + a) = \cos(x - a)$, find x. *Ans.* $x = 45^\circ$.

33. If $\sin(a - x) = \cos(a + x)$, find x. *Ans.* $x = -45^\circ$.

34. If $\tan\theta + \cot\theta = 2$, then $\sin\theta + \cos\theta = \sqrt{2}$.

EXAMPLES. (C).

1. If $\operatorname{cosec} a = 1.\dot{3}$, find $\sin a$, $\cos a$, $\tan a$, $\cot a$, $\sec a$.

Prove the truth of the following equations:

2. $(\sin^2 a - \sin^2 a \sin^2\beta)(\cos^2 a - \cos^2 a \cos^2\beta)$
$= (\sin^2\beta - \sin^2 a \sin^2\beta)(\cos^2\beta - \cos^2 a \cos^2\beta)$.

3. $\sec^2\alpha\tan^2\alpha(\sec^2\beta-1)(\tan^2\beta+1)=\sec^2\beta\tan^2\beta$
$(\sec^2\alpha-1)(\tan^2\alpha+1)$.

4. $\sin\alpha(1+\tan\alpha)+\cos\alpha(1+\cot\alpha)=\sec\alpha+\operatorname{cosec}\alpha$.

5. $\sin^{-1}\frac{p}{q}=\cot^{-1}\frac{\surd(q^2-p^2)}{p}$.

6. $\sin^{-1}\frac{p-q}{p+q}=\tan^{-1}\frac{p-q}{2\surd(pq)}$.

7. $\sec^2\alpha\operatorname{cosec}^2\alpha-2=\tan^2\alpha+\cot^2\alpha$.

8. $\sin\left(\frac{4n+3}{2}180^\circ+\alpha\right)=-\cos\alpha$.

9. $\cos\{(4n+2)90^\circ\pm\alpha\}=-\cos\alpha$.

10. If $\sec\theta\tan\theta=2\surd3$, then $\theta=60^\circ$.

11. If $\tan\theta\tan\phi=1$, then $\sec\theta=\operatorname{cosec}\phi$.

12. If $\frac{\sin\alpha}{\sin\beta}=\surd2$, $\frac{\tan\alpha}{\tan\beta}=\surd3$, then $\alpha=45^\circ$, $\beta=30^\circ$.

13. If $\frac{\cos\theta}{1+\sin\theta}+\tan\theta=2$, $\theta=60^\circ$.

14. If $\cos\phi=n\sin\alpha$, $\cot\phi=\frac{\sin\alpha}{\tan\beta}$,

then $$\cos\beta=\frac{n}{\surd(1+n^2\cos^2\alpha)}.$$

15. If $\cos\alpha+\cos(\beta-\gamma)=0$, then $\cos\beta+\cos(\alpha-\gamma)=0$,
and $\cos\gamma+\cos(\alpha-\beta)=0$.

16. If $\frac{\tan\alpha}{\sin\theta}-\frac{\tan\beta}{\tan\theta}=\sqrt{\tan^2\alpha-\tan^2\beta}$, then $\cos\theta=\frac{\tan\beta}{\tan\alpha}$.

17. If $\cos\alpha=\cos\beta\cos\gamma\pm\sin\beta\sin\gamma\cos\alpha$,

then $$\cos\beta=\cos\alpha\cos\gamma\pm\sin\alpha\sin\gamma\cos\beta.$$

18. If $\tan\beta=\sec\alpha\sec\gamma+\tan\alpha\tan\gamma$,

then $$\frac{\tan\alpha+\tan\beta\tan\gamma}{\tan\gamma+\tan\beta\tan\alpha}=\frac{\cos\alpha}{\cos\gamma}.$$

19. Prove that

$$\{(x-y)^2 \sin^2\theta - y^2\}^2 \cos^2\theta + \{(x-y)^2 \cos^2\theta - x^2\}^2 \sin^2\theta$$
$$= \{x^2 \sin^2\theta + y^2 \cos^2\theta\}^2.$$

20. Construct the angle whose secant is $(\sqrt{5}-1)$.

21. Given a the difference of the lengths of the shadows of a vertical stick when the sun's altitude is α, β, respectively; find the length of the stick.

If the length of the stick is a mean proportional between the lengths of the shadows, shew that α, β are complementary.

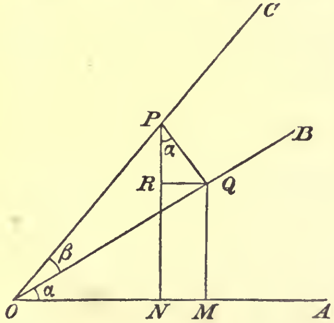
22. The elevation of Cader Idris at a point in the valley near Dolgelly is $\cot^{-1} 6$; at Ty Gwyn, $3\frac{1}{2}$ miles down the valley, it has the same elevation; at a point half-way between its elevation is $\cot^{-1} 5$. Its height above the valley is $\dfrac{7}{4\sqrt{11}}$ miles.

SECTION III.

TRIGONOMETRICAL RATIOS OF THE SUMS AND DIFFERENCES OF ANGLES, AND OF MULTIPLE ANGLES. EXPLANATION OF THE DOUBLE SIGN IN TRIGONOMETRICAL FORMULÆ.

27. We now proceed to determine the Trigonometrical Ratios of the sum and difference of any number of angles in terms of the ratios of the simple angles.

To find the Ratios of $(\alpha+\beta)$ in terms of those of α and β.



Let AOB be any angle α, BOC any angle β, then AOC is $(\alpha+\beta)$. In OC take any point P, and draw PN, PQ perpendicular to OA, OB respectively. From Q draw QM, QR perpendicular to OA, PN: then, since RPQ, RQO are each complementary to PQR, therefore $RPQ = RQO = QOM = \alpha$.

And we have,

$$\sin(\alpha+\beta) = \frac{PN}{OP} = \frac{QM}{OP} + \frac{PR}{OP}$$

$$= \frac{QM}{OQ} \cdot \frac{OQ}{OP} + \frac{PR}{PQ} \cdot \frac{PQ}{OP}.$$

Now
$$\frac{QM}{OQ} = \sin\alpha,\quad \frac{OQ}{OP} = \cos\beta,$$
$$\frac{PR}{PQ} = \cos\alpha,\quad \frac{PQ}{OP} = \sin\beta,$$

therefore $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$;

also
$$\cos(\alpha+\beta) = \frac{ON}{OP} = \frac{OM}{OP} - \frac{QR}{OP}$$
$$= \frac{OM}{OQ}\cdot\frac{OQ}{OP} - \frac{QR}{PQ}\cdot\frac{PQ}{OP}$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta;$$

also
$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta};$$

divide numerator and denominator of this fraction by $\cos\alpha\cos\beta$, and we have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta};$$

this might have been found directly by geometry as in the case of the sine and cosine.

The sine, cosine, and tangent being found as above, we may find the secant, cosecant, and cotangent.

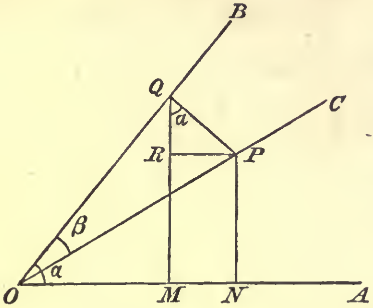
Put $\beta+\gamma$ for β in the above form, and we have

$$\sin(\alpha+\beta+\gamma) = \sin\alpha\cos(\beta+\gamma) + \cos\alpha\sin(\beta+\gamma)$$
$$= \sin\alpha(\cos\beta\cos\gamma - \sin\beta\sin\gamma) + \cos\alpha(\sin\beta\cos\gamma + \cos\beta\sin\gamma)$$
$$= \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\alpha\cos\gamma + \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma.$$

Similarly the forms for $\cos(\alpha+\beta+\gamma)$, $\tan(\alpha+\beta+\gamma)$ may be found; and the method may be extended to the sum of any number of angles.

28. To find the Trigonometrical Ratios of $(\alpha-\beta)$, in terms of those of α and β.

Let AOB be any angle α, BOC any angle β, then AOC is $(\alpha-\beta)$. In OC take any point P, and draw PN, PQ perpendicular to OA, OB respectively; from Q draw QM perpendicular to OA, and from P, PR perpendicular to QM: then $PQR = QOA = \alpha$, for PQR, QOA are each complementary to OQM.



And we have,

$$\sin(\alpha - \beta) = \frac{PN}{OP} = \frac{QM}{OP} - \frac{QR}{OP}$$

$$= \frac{QM}{OQ} \cdot \frac{OQ}{OP} - \frac{QR}{PQ} \cdot \frac{PQ}{OP}$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta;$$

also,

$$\cos(\alpha - \beta) = \frac{ON}{OP} = \frac{OM}{OP} + \frac{PR}{OP}$$

$$= \frac{OM}{OQ} \cdot \frac{OQ}{OP} + \frac{PR}{PQ} \cdot \frac{PQ}{OP}$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta;$$

also

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta};$$

and by means of these formulæ the Trigonometrical Ratios of the sums and differences of any number of angles may be calculated.

29. The figures in the above proofs have been drawn so that all the angles are less than 90°. It is left as practice to the student to adapt the proof to any case whatever, and to satisfy himself that the forms proved are true for any angles, a logical

proof of which he will find in more advanced treatises on the subject.

Assuming the form for $\sin(\alpha+\beta)$ to be true for any values of α and β, the forms for the other ratios might be deduced as follows:

$$\cos(\alpha+\beta)=\sin\{90-(\alpha+\beta)\}=\sin(90+\alpha+\beta) \qquad \text{(Art. 15)}$$
$$=\sin(90+\alpha)\cos\beta+\cos(90+\alpha)\sin\beta$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta, \text{ as before:}$$

again, putting $-\beta$ for β,

we have $$\sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$
$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta;$$

also $$\cos(\alpha-\beta)=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)$$
$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta;$$

since $\sin(-\beta)=-\sin\beta$, and $\cos(-\beta)=\cos\beta$. (Art. 16)

30. In the forms for $\alpha+\beta$, let $\alpha=\beta$,

then we have, $$\sin 2\alpha=2\sin\alpha\cos\alpha,$$
$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha,$$
$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha};$$

which give the Ratios of the double angles in terms of those of the single angles.

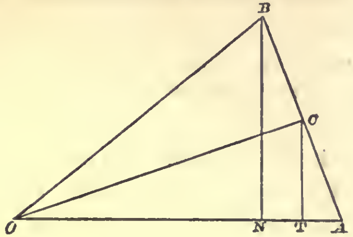
If we wish to determine $\sin 2\alpha$, $\cos 2\alpha$ in terms of $\sin\alpha$, or of $\cos\alpha$ only, we have

$$\sin 2\alpha=\pm 2\sin\alpha\sqrt{(1-\sin^2\alpha)},$$
$$\text{or }=\pm 2\cos\alpha\sqrt{(1-\cos^2\alpha)},$$
$$\cos 2\alpha=1-2\sin^2\alpha,$$
$$\text{or }=2\cos^2\alpha-1.$$

These forms may be found directly thus:

Let $AOC=BOC=\alpha$ make BCA perpendicular to OC, and BN, CT, to OA. Then

$$ABN=\alpha,\ BC=CA,\ NT=TA,\ BN=2CT,$$



and we have

$$\sin 2\alpha = \frac{BN}{OB} = 2 \cdot \frac{BN}{AB} \cdot \frac{BC}{OB} = 2 \cos\alpha \sin\alpha\,;$$

also,
$$\cos 2\alpha = \surd(1 - \sin^2 2\alpha) = \surd(1 - 4\sin^2\alpha\cos^2\alpha)$$
$$= \surd(1 - 4\sin^2\alpha + 4\sin^4\alpha) = 1 - 2\sin^2\alpha,$$

or directly from the figure, thus,

$$\cos 2\alpha = \frac{ON}{OB} = 1 - \frac{AN}{OB} = 1 - 2 \cdot \frac{AN}{AB} \cdot \frac{BC}{OB}$$
$$= 1 - 2\sin^2\alpha,$$

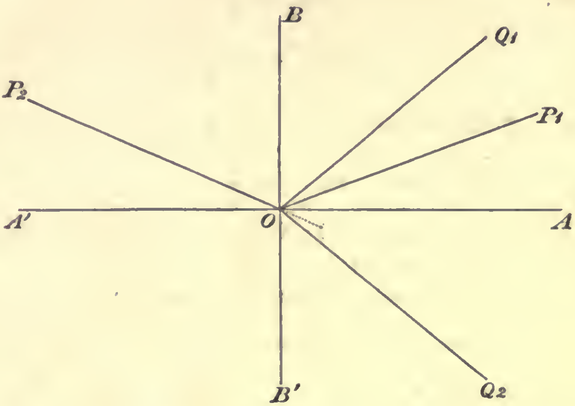
or thus,

$$\cos 2\alpha = \frac{ON}{OB} = \frac{OT}{OB} - \frac{NT}{OB} = \frac{OT}{OC} \cdot \frac{OC}{OB} - \frac{AT}{AC} \cdot \frac{AC}{OA}$$
$$= \cos^2\alpha - \sin^2\alpha\,;$$

also
$$\tan 2\alpha = \frac{BN}{ON} = \frac{2CT}{OT - NT} = \frac{2 \cdot \dfrac{CT}{OT}}{1 - \dfrac{AT}{CT} \cdot \dfrac{CT}{OT}}$$
$$= \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$

31. The double sign in the form for $\sin 2\alpha$ may be explained as follows:

Let AOP_1 be the smallest angle whose sine is equal to $\sin\alpha$: then we cannot tell whether the $\sin\alpha$, from which we are to determine $\sin 2\alpha$, corresponds to the bounding line OP_1 or OP_2, and therefore we ought not, if our formulæ have their proper generality, to be able to tell whether $\sin 2a$ corresponds to AOQ_1 or AOQ_2 (i.e. $180° + A'OQ_2$).



Now since $AOQ_2 = 2(180° - AOP_1) = 360° - AOQ_1$, therefore the sine of AOQ_2 must be equal to the sine of AOQ_1, and be of opposite sign, in accordance with the formulæ proved above. In the same way, if we consider the negative angles bounded by OP_1 and OP_2 we shall find that our formulæ must correspond to the bounding lines OQ_1, OQ_2, and be correct. So also if we suppose the revolving line to have completed any number of revolutions before reaching the positions OP_1, OP_2.

Precisely similar remarks will apply to $\sin 2a$ when determined in terms of $\cos a$.

The student is also recommended to examine why $\cos 2a$, $\tan 2a$ have not the double sign.

32. Putting $\beta = 2a$ in the forms for $(a + \beta)$, we have

$$\begin{aligned}\sin 3a &= \sin a \cos 2a + \cos a \sin 2a \\ &= \sin a\,(1 - 2\sin^2 a) + 2\cos^2 a \sin a \\ &= \sin a - 2\sin^3 a + 2\,(1 - \sin^2 a)\sin a \\ &= 3\sin a - 4\sin^3 a;\end{aligned}$$

so also $$\cos 3a = 4\cos^3 a - 3\cos a.$$

And similarly we may find the Ratios of any multiple angle.

33. If in the forms for $\sin 2\alpha$ we write $\frac{\alpha}{2}$ for α,

we have, $$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2},$$

also $$1 = \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}.$$

Hence, $$\pm\sqrt{(1+\sin\alpha)} = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \quad \text{..................... ..(A).}$$

$$\pm\sqrt{(1-\sin\alpha)} = \cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} \quad \text{............. ....(B).}$$

These formulæ are very useful; the following, which are deduced from them, not so much so.

Adding and subtracting (A) and (B), we have

$$2\cos\frac{\alpha}{2} = \pm\sqrt{(1+\sin\alpha)} \pm\sqrt{(1-\sin\alpha)},$$

$$2\sin\frac{\alpha}{2} = \pm\sqrt{(1+\sin\alpha)} \mp\sqrt{(1-\sin\alpha)}.$$

Each of these expressions is capable of four values, two and two of opposite signs, according to the signs of the radicals: the reason of which may be explained in a manner similar to that in Art. 31. In choosing practically which of the signs must be used in any particular case, we must return to the formulæ (A) and (B), and inquire, what the signs of $\cos\frac{\alpha}{2}$, $\sin\frac{\alpha}{2}$ must be, and which is the greatest: we can then determine whether $\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}$ is positive or negative, and so also, whether $\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}$ is positive or negative. Having thus given their proper signs to the radicals, we can add and subtract the equations (A) and (B), and obtain the proper values of $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$.

Ex. Let $\alpha = 296°$. Then $\cos 148°$ is negative, and numerically greater than $\sin 148°$, which is positive.

Hence $$\cos 148° + \sin 148° = -\sqrt{(1+\sin 296°)},$$
$$\cos 148° - \sin 148° = -\sqrt{(1-\sin 296°)},$$

whence the proper signs may be given to $\cos 148°$, $\sin 148°$.

34. Writing $\frac{a}{2}$ for a in the forms for $\cos 2a$, we obtain

$$\cos a = \cos^2 \frac{a}{2} - \sin^2 \frac{a}{2};$$

also $$\cos a = 2\cos^2 \frac{a}{2} - 1,$$

and therefore $$\cos \frac{a}{2} = \pm \sqrt{\left(\frac{1+\cos a}{2}\right)},$$

and $$\cos a = 1 - 2\sin^2 \frac{a}{2},$$

and therefore $$\sin \frac{a}{2} = \pm \sqrt{\left(\frac{1-\cos a}{2}\right)},$$

which give $\cos \frac{a}{2}$, $\sin \frac{a}{2}$ in terms of $\cos a$.

We may also find $\tan \frac{a}{2}$ in terms of $\tan a$, but the formula which results is of very little use.

35. Returning to the formulæ in Arts. 27, 28,

$$\sin(a+\beta) = \sin a \cos \beta + \cos a \sin \beta,$$
$$\sin(a-\beta) = \sin a \cos \beta - \cos a \sin \beta,$$
$$\cos(a+\beta) = \cos a \cos \beta - \sin a \sin \beta,$$
$$\cos(a-\beta) = \cos a \cos \beta + \sin a \sin \beta,$$

we get by addition and subtraction

$$2\sin a \cos \beta = \sin(a+\beta) + \sin(a-\beta),$$
$$2\cos a \sin \beta = \sin(a+\beta) - \sin(a-\beta),$$
$$2\cos a \cos \beta = \cos(a+\beta) + \cos(a-\beta),$$
$$2\sin a \sin \beta = \cos(a-\beta) - \cos(a+\beta),$$

which forms enable us when we have the product of two sines or cosines to replace them by the sum or difference of two other sines and cosines.

Again, in these forms let $\alpha = \frac{A+B}{2}$, and $\beta = \frac{A-B}{2}$, then $\alpha+\beta = A$, $\alpha-\beta = B$: substituting and transposing, we have

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2},$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2};$$

these forms enable us to pass from the sum or difference of two sines or cosines, to the product of two other sines or cosines.

35. *bis.* The following applications of the formulæ proved in this section are very important.

(1) $\sin(\alpha+\beta)\sin(\alpha-\beta)$

$$= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta$$
$$= \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta$$
$$= \sin^2\alpha - \sin^2\beta.$$

(2) Similarly $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$.

(3) $\tan(45^\circ \pm \theta) = \frac{\tan 45^\circ \pm \tan\theta}{1 \mp \tan 45^\circ \tan\theta} = \frac{1 \pm \tan\theta}{1 \mp \tan\theta}$.

(4) To find sin 18°.

Let $\theta = 18^\circ$, then $5\theta = 90^\circ$,

$\therefore\ 2\theta = 90^\circ - 3\theta$,

$\therefore\ \sin 2\theta = \cos 3\theta$,

$\therefore\ 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$,

$\therefore\ 2\sin\theta = 4\cos^2\theta - 3$

$\qquad\qquad = 1 - 4\sin^2\theta$,

$$\therefore\ \sin^2\theta + \frac{\sin\theta}{2} + \frac{1}{16} = \frac{1}{4} + \frac{1}{16} = \frac{5}{16},$$

$$\therefore\ \sin\theta = \frac{-1 \pm \sqrt{5}}{4},$$

$$\therefore\ \sin 18^\circ = \frac{\sqrt{5}-1}{4},$$

since it must be positive. The other value of $\sin\theta$ corresponds to the value $\theta = -54^\circ$, since in that case also $\sin 2\theta = \cos 3\theta$.

(5) Let $\sin^{-1} x = \theta$, $\sin^{-1} y = \phi$, then $\sin\theta = x$,

$$\cos\theta = \sqrt{(1-x^2)},\ \sin\phi = y,\ \cos\phi = \sqrt{(1-y^2)};$$

$$\therefore\ \sin(\theta+\phi) = x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)};$$

$$\therefore\ \sin^{-1} x + \sin^{-1} y = \theta + \phi = \sin^{-1}\{x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}\};$$

so also $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\{xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}$,

and $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$.

EXAMPLES. (A).

1. Find the sine of 15°.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Ans. $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

Apply the formulæ in Art. 27, to prove the following.

2. $\cos 15^\circ = \sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

3. $\sin 15^\circ = \cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

4. $\cot 15^\circ = \tan 75^\circ = 2 + \sqrt{3}$.

5. $\tan 15^\circ = \cot 75^\circ = 2 - \sqrt{3}$.

6. If $\sin\alpha = \frac{2}{3}$, $\sin\beta = \frac{3}{5}$, then $\sin(\alpha+\beta) = \frac{8+3\sqrt{5}}{15}$,

$$\cos(\alpha+\beta) = \frac{4\sqrt{5}-6}{15}.$$

7. If $\sin\alpha = \frac{3}{4}$, $\cos\beta = \frac{1}{3}$, then $\sin(\alpha+\beta) = \frac{3+2\sqrt{14}}{12}$,

$$\cos(\alpha+\beta) = \frac{\sqrt{7}-6\sqrt{2}}{12}.$$

8. If $\sin\alpha = \frac{5}{13}$, $\sin\beta = \frac{4}{5}$, then $\sin(45^\circ+\alpha+\beta) = \frac{79\sqrt{2}}{130}$.

9. Prove the values found in questions 2, 3, 4, 5 by means of Art. 28.

10. If $\sin\alpha = \frac{2}{3}$, $\sin\beta = \frac{3}{5}$, then $\sin(\alpha-\beta) = \frac{8-3\sqrt{5}}{15}$.

11. If $\sin\alpha = \frac{2}{3}$, then $\sin 2\alpha = \frac{4\sqrt{5}}{9}$, $\cos 2\alpha = \frac{1}{9}$.

12. Given $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ shew that $\sin 36^\circ = \sqrt{\frac{5-\sqrt{5}}{8}}$,

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4},\quad \sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

Prove the formulæ

13. $1 - \frac{1}{2}\sin 2\alpha \tan\alpha = \cos^2\alpha$.

14. $\sin 4\alpha = 4\cos 2\alpha \cos\alpha \sin\alpha$.

15. $\sin 8\alpha = 8\cos 4\alpha \cos 2\alpha \cos\alpha \sin\alpha$.

16. $\sin(\alpha+2\beta) = \sin\alpha - 2\sin\alpha\sin^2\beta + 2\cos\alpha\sin\beta\cos\beta$.

17. $\sin(\alpha+2\beta) = \sin\alpha + 2\sin\beta\cos(\alpha+\beta)$.

18. $\sin(\alpha-2\beta) = \sin\alpha - 2\sin\beta\cos(\alpha-\beta)$.

19. $\cos(\alpha+2\beta) = 2\cos\beta\cos(\alpha+\beta) - \cos\alpha$.

20. $\cos\alpha \pm \sin\alpha = \sqrt{1 \pm \sin 2\alpha}$.

21. $\cos^4\alpha - \sin^4\alpha = \cos 2\alpha$.

22. $\cos^6\alpha + \sin^6\alpha = 1 - \frac{3}{4}\sin^2 2\alpha = \frac{5 + 3\cos 4\alpha}{8}$.

23. $\cos^6\alpha - \sin^6\alpha = \cos 2\alpha\left(\frac{7 + \cos 4\alpha}{8}\right)$.

24. $\frac{\cos 3\alpha - \sin 3\alpha}{\cos\alpha + \sin\alpha} = 1 - 2\sin 2\alpha$.

25. $\frac{\cos 3\alpha + \sin 3\alpha}{\cos\alpha - \sin\alpha} = 1 + 2\sin 2\alpha$.

26. $\cos 6\alpha = \cos^2 3\alpha - \sin^2 3\alpha = \cos 2\alpha\,(2\cos 4\alpha - 1)$.

27. $\operatorname{cosec}^2\alpha - \sec^2\alpha = 4\cos 2\alpha \operatorname{cosec}^2 2\alpha$.

28. $\frac{\operatorname{cosec} 2\alpha - \cot 2\alpha}{\operatorname{cosec} 2\alpha + \cot 2\alpha} = \tan^2\alpha$.

29. $\sin 2\alpha + \cos\alpha = \frac{2\tan\alpha + \sec\alpha}{1 + \tan^2\alpha}$.

30. $\cot^2\alpha - \tan^2\alpha = \frac{8\cos 2\alpha}{1 - \cos 4\alpha}$.

31. $\frac{\cos\alpha - \cos 3\alpha}{\sin 3\alpha - \sin\alpha} = \tan 2\alpha$.

32. By means of Art. 33, find $\sin 75^\circ$, $\cos 75^\circ$, giving the proper signs to the radicals involved.

33. In Art. 33, give the proper signs to the radicals, (1) when α lies between 180° and 270°; (2) when α lies between 225° and 315°; (3) when α lies between 360° and 450°.

34. $\frac{1 + \sin\alpha}{\cos\alpha} = \frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}$.

35. $\frac{1 - \sin\alpha}{\cos\alpha} = \tan\left(45^\circ - \frac{\alpha}{2}\right)$.

36. $\frac{1 + \sec\alpha}{1 + \operatorname{cosec}\alpha} = \frac{2\tan\alpha}{\left(1 + \tan\frac{\alpha}{2}\right)^2}$.

37. $\tan\frac{\alpha}{2} + 2\sin^2\frac{\alpha}{2}\cot\alpha = \sin\alpha$.

38. By means of Art. 34, find $\sin 22\frac{1}{2}^\circ$, $\cos 22\frac{1}{2}^\circ$.

$$Ans.\quad \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}},\quad \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}.$$

39. Shew that $\sin 9^\circ = \sqrt{\dfrac{4-\sqrt{10+2\sqrt{5}}}{8}}$,

$$\cos 9^\circ = \sqrt{\frac{4+\sqrt{10+2\sqrt{5}}}{8}}.$$

40. Find expressions for $\sin 27^\circ$, $\sin 63^\circ$.

Apply the formulæ in Art. 35, to prove the following.

41. $2\sin a\cos 3a = \sin 4a - \sin 2a$.

42. $2\sin a\sin 3a = \cos 2a - \cos 4a$.

43. $2\sin 2a\cos a = \sin 3a + \sin a$.

44. $2\cos a\cos 2a = \cos 3a + \cos a$.

45. $\cos 3a\cos 4a = \frac{1}{2}\cos 7a + \frac{1}{2}\cos a$.

46. $4\sin a\sin 2a\sin 3a = \sin 2a + \sin 4a - \sin 6a$.

47. $4\cos a\cos 2a\cos 3a = 1 + \cos 2a + \cos 4a + \cos 6a$.

48. $4\cos a\cos 2a\sin 3a = \sin 2a + \sin 4a + \sin 6a$.

49. $8\sin a\sin 2a\sin 3a\sin 4a = 1 - \cos 6a - \cos 8a + \cos 10a$.

50. $\sin a + \sin 2a = 2\sin\frac{3a}{2}\cos\frac{a}{2}$.

51. $\sin 3a - \sin a = 2\cos 2a\sin a$.

52. $\cos 2a + \cos 3a = 2\cos\frac{5a}{2}\cos\frac{a}{2}$.

53. $\cos 3a - \cos 4a = 2\sin\frac{7a}{2}\sin\frac{a}{2}$.

54. $\sin\frac{3a}{2} + \sin 2a = 2\sin\frac{7a}{4}\cos\frac{a}{4}$.

55. $\sin 3a - \sin\frac{a}{2} = 2\cos\frac{7a}{4}\sin\frac{5a}{4}$.

56. $\sin a + \cos a = 2\sin 45^\circ\cos(45^\circ - a)$.

57. $\cos\alpha - \sin\alpha = \sqrt{2}\sin(45^\circ - \alpha)$.

58. $\cos\alpha + \cos 2\alpha + \cos 3\alpha = 4\cos\frac{\alpha}{2}\cos\alpha\cos\frac{3\alpha}{2} - 1$.

59. $4\cos\alpha\cos(120 - \alpha)\cos(120 + \alpha) = \cos 3\alpha$.

60. $\cos 9\alpha + 3\cos 7\alpha + 3\cos 5\alpha + \cos 3\alpha = 8\cos^3\alpha\cos 6\alpha$.

61. If $\tan\frac{\alpha}{2} = 2 - \sqrt{3}$, find α. *Ans.* $\alpha = 30^\circ$.

62. If $\cos 3\theta\cos\theta = \frac{5}{18}$, then $\sin\theta = \frac{1}{\sqrt{6}}$.

63. If $2\sin\theta\sin 36^\circ = 1$, then $\sin\theta = \sqrt{\frac{5 + \sqrt{5}}{10}}$.

EXAMPLES. (B).

1. Find the Trigonometrical Ratios of 81°, $52\frac{1}{2}^\circ$.

2. Find the Trigonometrical Ratios of $7\frac{1}{2}^\circ$, 105°.

3. If $\alpha + \beta + \gamma = 180^\circ$,

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma;$$

we have
$$\tan(\alpha + \beta) = \tan(180^\circ - \gamma),$$

or
$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma,$$

or
$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma. \quad \text{Q. E. D.}$$

Prove the formulæ,

4. $\tan(\alpha + 2\beta) = \frac{\tan\alpha + 2\tan\beta - \tan\alpha\tan^2\beta}{1 - 2\tan\alpha\tan\beta - \tan^2\beta}$.

5. $\frac{\tan 3\alpha}{\tan\alpha} = \frac{3 - \tan^2\alpha}{1 - 3\tan^2\alpha}$.

6. $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$.

7. $\cos(\alpha + \beta)\sin(\alpha - \beta) = \sin\alpha\cos\alpha - \sin\beta\cos\beta$.

8. $\sin(\alpha - \beta)\sin\gamma + \sin(\beta - \gamma)\sin\alpha + \sin(\gamma - \alpha)\sin\beta = 0$.

9. $\sin(\alpha + \beta)\cos\alpha - \cos(\alpha + \beta)\sin\alpha = \sin\beta$.

10. $\cos\alpha + \cos(\alpha + 2\beta) = 2\cos(\alpha + \beta)\cos\beta.$

11. $\cos\beta\cos(2\alpha + \beta) = \cos^2(\alpha + \beta) - \sin^2\alpha.$

12. $\dfrac{\sin(\alpha + \beta)\sin(\alpha - \beta)}{\cos^2\alpha\cos^2\beta} = \tan^2\alpha - \tan^2\beta.$

13. $\sin(45^0 + \alpha) = \dfrac{1}{\sqrt{2}}(\sin\alpha + \cos\alpha) = \cos(45^0 - \alpha).$

14. $\sin(60^0 + \alpha) = \dfrac{1}{2}(\sin\alpha + \sqrt{3}\cos\alpha) = \cos(30^0 - \alpha).$

15. $\sin(30^0 + \alpha) + \sin(30^0 - \alpha) = \cos\alpha.$

16. $\sec(\theta \pm \phi) = \dfrac{\sec\theta\sec\phi}{1 \mp \tan\theta\tan\phi}.$

17. $\dfrac{1 - \tan(45^0 - \theta)}{1 + \tan(45^0 - \theta)} = \tan\theta.$

18. $\cos^2(\alpha - \beta) - \sin^2(\alpha + \beta) = \cos 2\alpha\cos 2\beta.$

19. $\sin^2 24^0 - \sin^2 6^0 = \dfrac{\sqrt{5} - 1}{8}.$

20. $(\sin\theta - \sin\phi)^2 + (\cos\theta - \cos\phi)^2 = 2\,\text{vers}(\theta - \phi).$

21. $\cos(30 - \alpha) - \cos(30 + \alpha) = \sin\alpha.$

22. $\cot\alpha + \tan\alpha = 2\,\text{cosec}\,2\alpha.$

23. $\cot\alpha - \tan\alpha = 2\cot 2\alpha.$

24. $\dfrac{\sec\alpha - 1}{\sec\alpha + 1} = \tan^2\dfrac{\alpha}{2}.$

25. $\dfrac{\sin\alpha + \sin 3\alpha}{\cos\alpha + \cos 3\alpha} = \tan 2\alpha.$

26. $\tan\alpha + \sec\alpha = \tan\left(45^0 + \dfrac{\alpha}{2}\right).$

27. $2(1 - \cot 2\alpha\tan\alpha) = \sec^2\alpha.$

28. $\dfrac{1 + \sin\alpha}{1 + \cos\alpha} = \dfrac{1}{2}\left(1 + \tan\dfrac{\alpha}{2}\right)^2.$

29. $\dfrac{1 + \sin\alpha}{1 - \sin\alpha} = \tan^2\left(45^0 + \dfrac{\alpha}{2}\right).$

30. $\dfrac{\sin\alpha + \sin 2\alpha}{\cos\alpha - \cos 2\alpha} = \cot\dfrac{\alpha}{2}.$

31. $\tan\theta - \tan\frac{\theta}{2} = \tan\frac{\theta}{2}\sec\theta.$

32. $\frac{2\sin 2\alpha - \sin 4\alpha}{2\sin 2\alpha + \sin 4\alpha} + 1 = \sec^2\alpha.$

33. $\cos^2(\theta+\phi) + \cos^2(\theta-\phi) - \cos 2\theta\cos 2\phi = 1.$

34. $\cot^2\alpha - \tan^2\alpha = 4\operatorname{cosec} 2\alpha\cot 2\alpha.$

35. $\sin\alpha\tan\alpha + \cos\alpha\cot\alpha = (\sin\alpha + \cos\alpha)(2\operatorname{cosec} 2\alpha - 1).$

36. $\frac{\cos n\alpha - \cos(n+2)\alpha}{\sin(n+2)\alpha - \sin n\alpha} = \tan(n+1)\alpha.$

37. $\sin^4\theta + 2\cos\alpha\sin^2\theta\cos^2\theta + \cos^4\theta = 1 - \left(\sin\frac{\alpha}{2}\sin 2\theta\right)^2.$

38. $\sin 4\theta\tan^4\theta + 4\tan^3\theta + 2\sin 4\theta\tan^2\theta - 4\tan\theta + \sin 4\theta = 0.$

39. $\frac{\cos 3\alpha - 2\cos\alpha}{\sin 3\alpha + 2\sin\alpha}\tan\alpha = \frac{2\cos 2\alpha - 3}{2\cos 2\alpha + 3}.$

40. $\frac{\tan\frac{\alpha}{2} + \cot\frac{\alpha}{2} + 2}{\tan\frac{\alpha}{2} + \cot\frac{\alpha}{2} - 2} = \frac{\sin^2\left(45^\circ + \frac{\alpha}{2}\right)}{\sin^2\left(45^\circ - \frac{\alpha}{2}\right)}.$

41. $\cos\alpha + \cos(\alpha + 120^\circ) + \cos(\alpha - 120^\circ) = 0.$

42. $\cos^2\alpha + \cos^2(\alpha + 120^\circ) + \cos^2(\alpha - 120^\circ) = \frac{3}{2}.$

43. $\sin 3\alpha + \cos 3\alpha = \sqrt{2}\left\{\cos^2\frac{3\alpha - 45^\circ}{2} - \sin^2\frac{3\alpha - 45^\circ}{2}\right\}.$

44. $1 + \frac{1}{2}(\tan\alpha - \tan\beta)^2 + \frac{1}{2}\{\tan\alpha - \cot(\alpha+\beta)\}^2$

$+ \frac{1}{2}\{\tan\beta - \cot(\alpha+\beta)\}^2 = \tan^2\alpha + \tan^2\beta + \cot^2(\alpha+\beta).$

Solve the equations:

45. $\cos\theta + \cos 7\theta = \cos 4\theta.$ *Ans.* $\theta = 20^\circ$, or $\frac{45^\circ}{2}$.

46. $\cos\theta - \cos 3\theta = \sin 2\theta.$ *Ans.* $\theta = 0$, or 30°.

47. $\cos 4\theta + \cos 2\theta = \cos\theta.$ *Ans.* $\theta = 20^\circ$, or 90°.

48. $\sin 5x\cos 3x = \sin 9x\cos 7x.$ *Ans.* $x = 0$, or $7\frac{1}{2}^\circ$.

49. $\cos 2x + \sin x = 1.$ *Ans.* $x = 0$, or 30°.

50. $4 \cot 2\theta = \cot^2 \theta - \tan^2 \theta.$ *Ans.* $\theta = 0$, or 45°.

51. $\operatorname{cosec} \theta = 2 \cos 2\theta + 4 \cos^2 \theta.$ *Ans.* $\theta = 10^\circ$.

52. $\sin x + \sqrt{3} \cos x = \sqrt{2}.$ *Ans.* $x = 15^\circ$, or 75°.

53. $\dfrac{1}{\sin x} + \dfrac{\sqrt{3}}{\cos x} = 4.$ *Ans.* $x = 30^\circ$.

54. $3(\sin^4 \theta - \cos^4 \theta) + 4 \cos^4 \theta = \cos^3 2\theta.$ *Ans.* $\theta = 0$.

55. $\sin 9x + \sin 5x + 2 \sin^2 x = 1.$ *Ans.* $x = 45^\circ$, or $2\frac{1}{7}^\circ$.

56. $\cos 4\theta + \cos 2\theta + \cos \theta = 0.$ *Ans.* $\theta = 90^\circ$, or 40°.

57. $\cos ax \cos bx = \cos (a + c) x \cos (b + c) x.$

Ans. $x = 0$, or $\dfrac{180^\circ}{c}$, or $\dfrac{180^\circ}{a + b + c}$.

58. If $\sin x = \sin \alpha \sin (x + y)$, then $\tan x = \dfrac{\sin \alpha \sin y}{1 - \sin \alpha \cos y}$.

59. If $\tan \beta = \dfrac{2 \sin \alpha \sin \gamma}{\sin (\alpha + \gamma)}$, then $\cot \alpha$, $\cot \beta$, $\cot \gamma$ are in A.P.

60. If $\sin \beta = m \sin (2\alpha + \beta)$, then $\tan (\alpha + \beta) = \dfrac{1 + m}{1 - m} \tan \alpha$.

61. If $\cot \theta = \dfrac{a}{b}$, then $a \sin \theta - b \cos \theta = 0$;

and $\dfrac{a \cos \theta - b \sin \theta}{a \cos \theta + b \sin \theta} = \dfrac{a^2 - b^2}{a^2 + b^2}$.

62. If $\sin x = \dfrac{\sin \alpha}{\sin \gamma}$, $\sin (90^\circ - x) = \dfrac{\sin \beta}{\sin \gamma}$,

then $\cos^2 \alpha + \cos^2 \beta + \sin^2 \gamma = 2$.

63. If $a \cos \phi = b \cos \theta$, then $\cot \dfrac{\phi + \theta}{2} \cot \dfrac{\phi - \theta}{2} = \dfrac{a + b}{a - b}$.

64. If $\tan \beta = \sec \alpha \sec \gamma + \tan \alpha \tan \gamma$, then $\cos 2\beta$ is negative.

65. If $\tan (\alpha + \theta) \cot \alpha = \tan (\beta + \phi) \cot \beta$,

then $\sin (2\alpha + \theta) \operatorname{cosec} \theta = \sin (2\beta + \phi) \operatorname{cosec} \phi$.

66. If $\cos \theta = \tan \beta \cot \gamma$, $\cot \theta = \sin \beta \cot \alpha$,

then $\cos \gamma = \cos \alpha \cos \beta$.

67. If $a\sin\alpha + x\cos\theta = b + x\tan\beta\sin\theta$,
and $a\cos\alpha - x\sin\theta = x\tan\beta\cos\theta$,

$$\text{then } \tan\theta = \frac{a\cos(\beta-\alpha) - b\sin\beta}{a\sin(\beta-\alpha) + b\cos\beta}.$$

68. If $\tan(\phi+\alpha)$, $\tan\phi$, $\tan(\phi+\beta)$ are in A. P. so also are $\cot\alpha$, $\tan\phi$, $\cot\beta$.

Prove the formulæ,

69. $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = 90°$.

70. $\cot^{-1}a - \cot^{-1}b = \cot^{-1}\frac{1+ab}{b-a}$.

71. $\text{vers}^{-1}\frac{4a-x}{a} = 180° - \text{vers}^{-1}\frac{x-2a}{a}$.

72. $\tan^{-1}\frac{3}{5} + \cot^{-1}\frac{7}{3} = \cot^{-1}\frac{13}{18}$.

73. $\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}5$.

74. $45° - 3\tan^{-1}\frac{1}{4} - \tan^{-1}\frac{1}{20} = \tan^{-1}\frac{7}{467}$.

75. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = 90°$.

76. $\tan^{-1}\frac{1}{x-1} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{x^2-x+1}$.

77. $\tan(\sin^{-1}m + \cos^{-1}n) = \dfrac{mn + \sqrt{(1-m^2)(1-n^2)}}{n\sqrt{1-m^2} - m\sqrt{1-n^2}}$.

78. If $\tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2} = 30°$, then $x = -1 \pm \sqrt{2\sqrt{3}}$.

79. If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = 45°$, then $x = \pm\frac{1}{\sqrt{2}}$.

EXAMPLES. (C).

Prove the formulæ,

1. $\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\tan 2\alpha+\sec 2\alpha.$

2. $\dfrac{\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)}{\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)}=\tan(\alpha+\beta).$

3. $16\operatorname{cosec}^2 4\alpha-4\sec^2 2\alpha=\sec^2\alpha\operatorname{cosec}^2\alpha.$

4. $\dfrac{\sin\alpha+\sin 2\alpha}{1+\cos\alpha+\cos 2\alpha}=\tan\alpha.$

5. $\dfrac{\cos(\alpha+45^0)}{\cos(\alpha-45^0)}=\sec 2\alpha-\tan 2\alpha.$

6. $4\sin\dfrac{\theta}{3}\sin\dfrac{180-\theta}{3}\sin\dfrac{180+\theta}{3}=\sin\theta.$

7. $\dfrac{\tan\alpha+\sec\alpha}{\cot\alpha+\operatorname{cosec}\alpha}=\tan\left(45^0+\dfrac{\alpha}{2}\right)\tan\dfrac{\alpha}{2}.$

8. $\dfrac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}=\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma.$

9. $\tan 4\alpha=\dfrac{4\tan\alpha-4\tan^3\alpha}{1-6\tan^2\alpha+\tan^4\alpha}.$

10. $\tan(30^0+\alpha)\tan(30^0-\alpha)=\dfrac{2\cos 2\alpha-1}{2\cos 2\alpha+1}.$

11. $\dfrac{\sin^2 2\alpha-4\sin^2\alpha}{\sin^2 2\alpha+4\sin^2\alpha-4}=\tan^4\alpha.$

12. $2\operatorname{vers}\dfrac{180^0+\alpha}{2}\operatorname{vers}\dfrac{180^0-\alpha}{2}=\operatorname{vers}(180^0-\alpha).$

13. $\tan\alpha\tan(\alpha-\beta)(\cot\alpha+\tan\beta)=\tan\alpha-\tan\beta.$

14. $\dfrac{1+\operatorname{cosec}^2\alpha\tan^2\theta}{1+\operatorname{cosec}^2\beta\tan^2\theta}=\dfrac{1+\cot^2\alpha\sin^2\theta}{1+\cot^2\beta\sin^2\theta}.$

15. $\sin^2\beta+\sin^2(\alpha-\beta)+2\sin\beta\sin(\alpha-\beta)\cos\alpha=\sin^2\alpha.$

16. $\dfrac{\sin^2\alpha-\sin^2\beta}{\sin\alpha\cos\alpha-\sin\beta\cos\beta}=\tan(\alpha+\beta).$

17. $1 + \tan\alpha \tan\frac{\alpha}{2} = \sec\alpha.$

18. $$\left(\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2}\right)\left(\sin\frac{\alpha-\beta}{2} + \cos\frac{\alpha-\beta}{2}\right) = \sin\alpha + \cos\beta.$$

19. $\cos^2(\alpha-\theta) + \cos^2\theta - 2\cos(\alpha-\theta)\cos\theta\cos\alpha = \sin^2\alpha.$

20. $\tan(45^\circ+\theta) - \tan(45^\circ-\theta) = \dfrac{2(\sec 2\theta - \cos 2\theta)}{\sin 2\theta}.$

21. $\tan\frac{\theta}{3}\cot\frac{1}{3}(90^\circ-\theta)\cot\frac{1}{3}(90^\circ+\theta) = \tan\theta.$

22. $$\frac{\tan\phi - \cos\theta}{\tan\phi + \cos\theta} = \frac{1 - \tan^2\frac{\theta}{2}\tan(45^\circ+\phi)}{\tan^2\frac{\theta}{2} - \tan(45^\circ+\phi)}.$$

23. $(\cos\theta + \sin\theta)^3 = \frac{3}{2}(\cos\theta + \sin\theta) - \frac{1}{2}(\cos 3\theta - \sin 3\theta).$

24. $64\sin^3\theta\cos^4\theta = 3\sin\theta + 3\sin 3\theta - \sin 5\theta - \sin 7\theta.$

25. $\dfrac{\cos(45^\circ+\alpha) - \cos(45^\circ+3\alpha)}{\cos(45^\circ-\alpha) + \cos(45^\circ-3\alpha)} = \tan\alpha.$

26. $\cos 3\theta\sin^3\theta + \sin 3\theta\cos^3\theta = \frac{3}{4}\sin 4\theta.$

27. $\sin 3\theta\sin^3\theta + \cos 3\theta\cos^3\theta = \cos^3 2\theta.$

28. $$\frac{(\sec\theta\sec\phi + \tan\theta\tan\phi)^2 - (\tan\theta\sec\phi + \sec\theta\tan\phi)^2}{2(1+\tan^2\theta\tan^2\phi) - \sec^2\theta\sec^2\phi} = \frac{\sec 2\theta\sec 2\phi}{\sec^2\theta\sec^2\phi}.$$

29. $$(\sec\theta + \operatorname{cosec}\theta + \sec\theta\operatorname{cosec}\theta)\left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{4}\right) = \left(\sec\frac{\theta}{2} + \operatorname{cosec}\frac{\theta}{2}\right)\sec^2\frac{\theta}{4}.$$

30. $\tan\theta + \tan(\theta+60^\circ) + \tan(\theta-60^\circ) = 3\tan 3\theta.$

31. $(2+\sin 2\theta)\sin(\theta-45^\circ) + (2-\sin 2\theta)\sin(\theta+45^\circ) = 2\sqrt{2}\sin^3\theta$

32. $\tan^{-1}\dfrac{1}{\sqrt{2}} + \sin^{-1}\dfrac{1}{\sqrt{2}} = \tan^{-1}\dfrac{\sqrt{2}+1}{\sqrt{2}-1}.$

Solve the equations,

33. $5 \tan x = \tan 3x.$ *Ans.* $\tan x = \pm \sqrt{\frac{1}{7}}$, or 0.

34. $4 \sin \theta \sin (\theta - \alpha) = 2 \cos \alpha - 1.$ *Ans.* $\theta = 30 + \frac{\alpha}{2}.$

35. $90^0 + \sin^{-1} x = \tan^{-1} x.$ *Ans.* $x = \sqrt{\frac{\sqrt{5} - 1}{2}}.$

36. $\tan^{-1} 2x + \tan^{-1} 3x = 45^0.$ *Ans.* $x = -1$, or $\frac{1}{6}.$

37. $\tan^{-1} \frac{a + x}{a} + \tan^{-1} \frac{a - x}{a} = 30^\circ.$ *Ans.* $x = a \sqrt{2\sqrt{3}}.$

38. $\tan^{-1} 3x + 2 \tan^{-1} 7x = 45^\circ.$ *Ans.* $x = \frac{1}{21}.$

39. $\tan^{-1} ax + \tan^{-1} \frac{x}{a} = 45^\circ.$

Ans. $x = \frac{-(a^2 + 1) \pm \sqrt{a^4 + 6a^2 + 1}}{2a}$

40. $\tan^{-1} (x + 1) \sqrt{2} - \tan^{-1} \frac{x - 1}{\sqrt{2}} = \cot^{-1} 4 \sqrt{2}.$

Ans. $x = 6$, or -2.

41. $\cos^{-1} x + \cos^{-1} (1 - x) = \cos^{-1} (-x).$ *Ans.* $x = 0$, or $\frac{1}{2}.$

42. $3 \tan^{-1} t = \tan^{-1} 5t.$ *Ans.* $t = 0$, or $\pm \frac{1}{\sqrt{7}}.$

43. $\cos^{-1} x + \cos^{-1} (1 - x) = \cos^{-1} \sqrt{(x - x^2)}.$

Ans. $x = 0$, or $\frac{1}{2}$, or 1.

44. $\tan^{-1} \frac{1}{x - 1} - \tan^{-1} \frac{1}{x + 1} = \tan^{-1} a.$

Ans. $x = \sqrt{\frac{2}{a}}.$

45. $\tan (\alpha + x) \tan (\alpha - x) = \frac{1 - 2 \cos 2\alpha}{1 + 2 \cos 2\alpha}.$ *Ans.* $x = 30^\circ.$

46. $\cos^{-1} \frac{1 - x^2}{1 + x^2} + \tan^{-1} \frac{2x}{1 - x^2} = 240^0.$

Ans. $x = \sqrt{3}$, or $-\frac{1}{\sqrt{3}}.$

47. $\{\cot(m+n)a+1\}\cos 2ma=\{\cot(m+n)a-1\}\cos 2na.$

Ans. $a=\dfrac{45^\circ}{m-n}.$

48. $\sec(180-a)\cos(90+a)\cos(180-a)$

$=\tan(90-a)\sin(270+a).$

Ans. $a=\pm 45^\circ.$

49. $\cos na+\cos(n-2)a=\cos a.$ *Ans.* $a=90^\circ$, or $\dfrac{60^\circ}{n-1}$,

50. $\operatorname{cosec}^2\dfrac{\theta}{2}-\sec^2\dfrac{\theta}{2}=2\sqrt{3}\operatorname{cosec}^2\theta.$ *Ans.* $\theta=30^\circ.$

51. $\sin\theta-\cos\theta=2\sqrt{2}\cos\theta\sin\theta.$

Ans. $\theta=15^\circ$, or 75°, or 135°.

52. $4\cos\theta\cos 2\theta=1.$ *Ans.* $\theta=120^\circ$, or 36°, or 108°.

53. $\sin^6 x+\cos^4 x=\dfrac{3}{8}.$ *Ans.* $x=45^\circ$, or $\sin^{-1}\sqrt{\dfrac{\sqrt{(29)}-3}{4}}.$

54. $\tan(a+\theta)\tan(a-\theta)=\tan^2 2a-\tan^2 2\theta.$

Ans. $\theta=\pm a$, or $\dfrac{1}{2}\cos^{-1}\left(\dfrac{-\cos 2a}{1\pm\cos 2a}\right).$

55. $\sec^2\dfrac{\theta}{2}\sec\theta\left(\cot^2\dfrac{\theta}{2}-\cot^2\dfrac{3\theta}{2}\right)=8\left(1-\cot^2\dfrac{3\theta}{2}\right).$

Ans. $\theta=60^\circ.$

56. $\tan\theta+\cot\theta=2\operatorname{cosec} a\,(\sqrt{1+\sin a}+\sqrt{1-\sin a}).$

Ans. $\theta=\dfrac{a}{4}$, or $45^\circ-\dfrac{a}{4}.$

57. $\sin(3x+a)+\cos(3x-a)=\cos(45^\circ+x).$

Ans. $x=45^\circ$, or $\dfrac{1}{2}\sin^{-1}\dfrac{1}{4}\{\sec(45^\circ-a)-2\}.$

58. $\dfrac{1}{\sqrt{1-\sin\theta}-1}+\dfrac{1}{\sqrt{1+\sin\theta}-1}=\dfrac{1}{\sin\theta}.$

Ans. $\theta=0$, 180°, or 60°.

59. $\sin(\tan^{-1}x)+\tan(\sin^{-1}x)=mx.$

$$Ans.\quad x=\frac{1}{m}\left(m^4-m^2-2\pm2\sqrt{2m^2+1}\right)^{\frac{1}{2}}.$$

60. If $\tan\beta+\cot\beta=2\sec 2\alpha$, then $\alpha+\beta=45^\circ$.

61. If $\tan(\theta+\alpha)\tan(\theta-\beta)=\tan^2\theta$,

$$\text{then }\tan 2\theta=\frac{2\sin\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

62. If $\tan 2\alpha=2\tan\beta$, and $\tan\gamma=\tan^3\alpha$,

shew that $\beta-\gamma=\alpha$.

63. Shew that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=45^\circ$.

64. Find $\tan(\alpha+\beta+\gamma)$ in terms of $\tan\alpha$, $\tan\beta$, $\tan\gamma$; and thence shew that if $\alpha+\beta+\gamma=180^\circ$,

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$$

65. If $\tan(\alpha+\theta)=n\tan(\alpha-\theta)$, then $\sin 2\theta=\frac{n-1}{n+1}\sin 2\alpha$.

66. If $\sin 2\theta=\tan\theta$, then $\text{chd}\,\theta=\sqrt{2-\sqrt{2}}$.

67. If $\sin\theta+\sin\phi=m$, $\cos\theta+\cos\phi=n$,

$$\text{then } m\sin\theta+n\cos\theta=\frac{m^2+n^2}{2}=m\sin\phi+n\cos\phi.$$

68. If $\tan^3\theta=\frac{b}{a}$, then $\frac{a}{\cos\theta}+\frac{b}{\sin\theta}=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.

69. Find $\cos(\alpha+\beta+\gamma)$, and if $\alpha+\beta+\gamma=180^\circ$, shew that $\cot\alpha+\cot\beta+\cot\gamma=\cot\alpha\cot\beta\cot\gamma+\text{cosec}\,\alpha\,\text{cosec}\,\beta\,\text{cosec}\,\gamma$.

If $\alpha+\beta+\gamma=180^\circ$, then

70. $\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.

71. $\cos\alpha+\cos\beta+\cos\gamma=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}+1$.

72. $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$.

73. $\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}=2\left(1+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)$.

74. $$\frac{\tan\alpha}{\tan\beta}+\frac{\tan\beta}{\tan\alpha}+\frac{\tan\alpha}{\tan\gamma}+\frac{\tan\gamma}{\tan\alpha}+\frac{\tan\beta}{\tan\gamma}+\frac{\tan\gamma}{\tan\alpha}=\sec\alpha\sec\beta\sec\gamma-2.$$

75. $$\tan\left(45^\circ-\frac{\alpha}{4}\right)\tan\left(45^\circ-\frac{\beta}{4}\right)\tan\left(45^\circ-\frac{\gamma}{4}\right)=\frac{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)^2}.$$

76. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\alpha\sin\beta\sin\gamma.$

77. $\cos 2\alpha+\cos 2\beta+\cos 2\gamma+4\cos\alpha\cos\beta\cos\gamma+1=0.$

78. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1.$

79. $$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}=4\cos\frac{\alpha+\beta}{4}\cos\frac{\beta+\gamma}{4}\cos\frac{\alpha+\gamma}{4}.$$

80. $$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=-4\cos\frac{3\alpha}{2}\cos\frac{3\beta}{2}\cos\frac{3\gamma}{2}.$$

81. $$\sin^3\alpha+\sin^3\beta+\sin^3\gamma=3\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+\cos\frac{3\alpha}{2}\cos\frac{3\beta}{2}\cos\frac{3\gamma}{2}.$$

82. If $\cos^{-1}\dfrac{x}{a}=2\sin^{-1}\dfrac{y}{a}$, then $ax+2y^2=a^2$.

83. If $\cot(\alpha-\gamma)-\cot(\alpha-\theta)=\cot(\alpha+\gamma)-\cot(\alpha+\theta)$,
then $\sin(\alpha-\gamma)\sin(\theta-\alpha)=\sin(\alpha+\gamma)\sin(\theta+\alpha)$.

84. If $\tan^2 x=\tan(\alpha-x)\tan(\alpha+x)$, then $\sin 2x=\sqrt{2}\,.\sin\alpha$.

85. If $\sin(\alpha-\beta)=\dfrac{1}{m}\sin(\alpha+\beta)$,

$$\text{then }\cot(\alpha+\beta)=\frac{1}{2}\left(\frac{m-1}{m}\cot\beta-\frac{m+1}{m}\tan\beta\right).$$

86. If $\cos\theta=\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$, then $\tan\dfrac{\theta}{2}=\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$.

87. If $\cot\theta=m\cot(\alpha-\theta)$,

$$\text{then }\theta=\frac{1}{2}\left\{\alpha-\sin^{-1}\left(\frac{m-1}{m+1}\sin\alpha\right)\right\}.$$

88. If $\tan\frac{\alpha}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\beta}{2}$, then $\cos\beta=\frac{e+\cos\alpha}{1+e\cos\alpha}$.

89. If $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)}=\cos 2\beta$, then $\tan(45^\circ-\theta)=\tan^2\beta\cot(45^\circ-\alpha)$.

90. If $2\cot\beta=\cot\alpha+\cot\gamma$, then $\cot(\beta-\alpha)+\cot(\beta-\gamma)=2\cot\beta$.

91. If $\cot^2\theta-\tan^2\theta=2\sqrt{3}\,.\sec 2\theta$, then $\sin 2\theta=\pm(\sqrt{3}-1)$.

92. If $\cos\theta\cot\phi=\cos\theta_1\cot\phi_1$,

$$\text{then } \cot\frac{\theta+\theta_1}{2}=\frac{\sin(\phi+\phi_1)}{\sin(\phi-\phi_1)}\tan\frac{\theta_1-\theta}{2}.$$

93. If $\alpha+\beta+\gamma=180^\circ$, then

$$\cot m\alpha\cot m\beta+\cot m\alpha\cot m\gamma+\cot m\beta\cot m\gamma=1,$$

where m is any integer.

94. If $\cos\theta+\cos(\theta-\beta)=\cos(\theta-\alpha)+\cos(\theta+\alpha-\beta)$, then each member of the equation is equal to $\cos\theta+\cos(\theta-\alpha)$, or 0.

95. If $\tan(45^\circ+\theta)\tan(45^\circ+\phi)\tan(45^\circ+\psi)=1$,

$$\text{then } \sin 2\theta\sin 2\phi\sin 2\psi+\sin 2\theta+\sin 2\phi+\sin 2\psi=0.$$

96. If $\alpha+\beta+\gamma=60^\circ$, then

$$1+2\cos(\alpha+\beta-\gamma)+2\cos(\alpha+\gamma-\beta)+2\cos(\beta+\gamma-\alpha)=8\cos\alpha\cos\beta\cos\gamma.$$

97. If $\theta+\phi=240^\circ$, and $\operatorname{vers}\theta=4\operatorname{vers}\phi$,

$$\text{then } \tan\frac{\theta}{2}=-\frac{\sqrt{3}}{2},\text{ or }\infty,\text{ and }\tan\frac{\phi}{2}=-\frac{\sqrt{3}}{5},\text{ or }\frac{1}{\sqrt{3}}.$$

98. $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$

$$+4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}=0.$$

99. $\sin(\alpha+\beta)\sin(\beta+\gamma)-\sin\alpha\sin\gamma=\sin(\alpha+\beta+\gamma)\sin\beta$.

100. $4\cos(\alpha+\beta+\gamma)\cos(\alpha+\beta-\gamma)\cos(\alpha-\beta+\gamma)\cos(\alpha-\beta-\gamma)$

$$=\cos^2 2\alpha+\cos^2 2\beta+\cos^2 2\gamma+2\cos 2\alpha\cos 2\beta\cos 2\gamma-1.$$

101. $\cos 2(\alpha+\beta+\gamma)\sin 3(\alpha+\beta)+\cos(\gamma-\alpha)\sin(\gamma-\beta)$

$$=\sin 2(\alpha+\beta+\gamma)\cos 3(\alpha+\beta)-\sin(\gamma-\alpha)\cos(\gamma-\beta).$$

102. $$\sin\left(\frac{\theta+\phi}{2}-\psi\right)\cos\frac{\theta+\psi}{2}-\sin\left(\frac{\theta+\psi}{2}-\phi\right)\cos\frac{\theta+\phi}{2}$$
$$=\sin\frac{\phi-\psi}{2}(\cos\theta+\cos\phi+\cos\psi).$$

103. $$\tan^{-1}\sqrt{\cos 60^\circ}+n\cos^{-1}(\tan^2 30^\circ)=\left(n+\frac{1}{2}\right)\cos^{-1}\frac{1}{3}.$$

104. $$(2^n \operatorname{cosec} 2^n\alpha)^2+(2^n \sec 2^n\alpha)^2=(2^{n+1}\operatorname{cosec} 2^{n+1}\alpha)^2.$$

105. $$2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\cdot\tan\frac{x}{2}\right\}=\cos^{-1}\left\{\frac{b+a\cos x}{a+b\cos x}\right\}.$$

106. $$\cot\{\theta+\tan^{-1}(\tan^3\theta)\}=2\cot 2\theta.$$

107. $$3\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{26}-45^\circ=\tan^{-1}\frac{1}{2057}.$$

108. $$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=45^\circ.$$

109. $$\cos^8\frac{45^\circ}{2}+\cos^8\frac{135^\circ}{2}+\cos^8\frac{225^\circ}{2}+\cos^8\frac{315^\circ}{2}=\frac{17}{16}.$$

110. If $x=r\sin\dfrac{\theta-a}{2}$, $y=r\sin\dfrac{\theta+a}{2}$,
$$\text{then } x^2-2xy\cos a+y^2=r^2\sin^2 a.$$

111. If $\cos\theta=\cot\beta\cot\gamma$, $\cos\phi=\cot a\cot\gamma$, $\cos\psi=\cot a\cot\beta$, and $\theta+\phi+\psi=180^\circ$, then $\cos^2 a+\cos^2\beta+\cos^2\gamma=1$.

Solve the equations:

112. $$\frac{x}{a}\cos\frac{\phi+\theta}{2}+\frac{y}{b}\sin\frac{\phi+\theta}{2}=\cos\frac{\phi-\theta}{2},\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Ans. $x=a\cos\theta$, $y=b\sin\theta$, or $x=a\cos\phi$, $y=b\sin\phi$.

113. $\sin^{-1}x+\sin^{-1}y=120^\circ$,
$\cos^{-1}x-\cos^{-1}y=60^\circ$.

Ans. $x=\pm 1$, $y=\pm\dfrac{1}{2}$.

114. If a, β, γ are the angles of a triangle, and
$$\cos\beta=\frac{\sin a}{2\sin\gamma},$$
then the triangle is isosceles.

115. An object 6 feet high, placed on the top of a tower, subtends an angle $\tan^{-1} .015$ at a place whose horizontal distance from the foot of the tower is 100 feet: find the height of the tower. *Ans.* 170 feet nearly.

116. A person observes the angle subtended by a tower and its spire to be the same; he knows the height h of the tower, and his distance a from it. The height of the spire is

$$\frac{a^2+h^2}{a^2-h^2}h.$$

117. A person at distance of 20 yards from the nearer of two towers in the same straight line with him, and 10 yards apart, observes them to subtend the same angle. Passing the nearer tower a certain distance, he observes them again subtend the same angle, the complement of the former. The heights of the towers are $6\sqrt{5}$, $4\sqrt{5}$ yards respectively.

118. A person at distance a from a tower, observes the elevation α of the tower, and of the top of the flagstaff $(90^\circ-\alpha)$; the height of flagstaff is $2a\cot 2\alpha$.

If the distance from the tower is unknown, and on receding c feet the elevation of tower is $\frac{\alpha}{2}$; then the height of flagstaff is $c \operatorname{cosec} \alpha \cos 2\alpha$.

119. From each of two stations in the same horizontal plane at distance D from each other, a pillar on a hill in the same vertical plane with the stations is observed to subtend the same angle at each of the two stations, and the elevations of the top of the pillar are ϵ, ϵ' at the two stations respectively. The height of pillar is

$$\frac{D\cos(\epsilon'+\epsilon)}{\sin(\epsilon'-\epsilon)}.$$

120. The length of the shadow of a tower, height h, is observed twice in the same day to be a, b, and the difference of the altitude of the sun on the two occasions was α, shew that

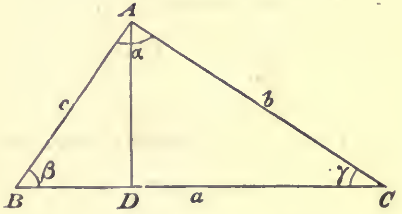
$$h^2+(a-b)h\cot\alpha+ab=0;$$

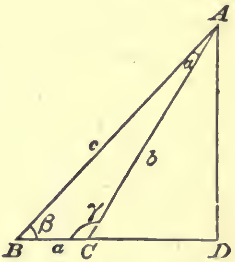
and explain the two values of h obtained from this equation.

SECTION IV.

RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE. THE RADII OF THE CIRCUMSCRIBED AND INSCRIBED CIRCLES.

36. We now proceed to establish some important relations between the sides and the Trigonometrical Ratios of the angles of a triangle.





Let ABC be a triangle, from A draw AD perpendicular to BC or BC produced: then denoting the angles of the triangle by α, β, γ, and the sides opposite to them by a, b, c respectively, we have

$$\sin\beta = \frac{AD}{AB}, \text{ or } AD = c\sin\beta;$$

$$\sin\gamma = \sin ACD = \frac{AD}{AC}, \text{ or } AD = b\sin\gamma.$$

Hence $$c\sin\beta = b\sin\gamma,$$

or $$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

Similarly, by dropping a perpendicular from B on AC, we have,

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c};$$

and therefore $$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

37. Again we have

$$\begin{aligned} BD &= AB\cos\beta = c\cos\beta, \\ CD &= AC\cos ACD = b\cos\gamma, \text{ if } \gamma \text{ is acute;} \\ &= -b\cos\gamma, \text{ if } \gamma \text{ is obtuse.} \end{aligned}$$

But $a = BC = BD + CD$ in the first case,

$= BD - CD$ in second case;

and therefore in both cases,

$$a = c\cos\beta + b\cos\gamma.$$

This may be derived from the forms in the preceding Article as follows;

since $\alpha = 180 - (\beta + \gamma)$,

$$c\sin\alpha = c\sin(\beta+\gamma) = c\sin\beta\cos\gamma + c\cos\beta\sin\gamma,$$

or $$a\sin\gamma = b\sin\gamma\cos\gamma + c\cos\beta\sin\gamma;$$

and therefore $$a = b\cos\gamma + c\cos\beta,$$

as before.

Similar forms may be proved for each of the other sides of the triangle.

38. Again $(a - b\cos\gamma)^2 = (c\cos\beta)^2$;

$$\begin{aligned} \therefore\ a^2 - 2ab\cos\gamma + b^2\cos^2\gamma &= c^2 - c^2\sin^2\beta \\ &= c^2 - b^2\sin^2\gamma; \end{aligned}$$

$$\therefore\ c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

and so for the other sides. This may also be put in the form

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}.$$

This formula may be obtained geometrically as follows.

By Euclid, II. 13, if ACB is acute,

$$AB^2 = BC^2 + AC^2 - 2BC \,.\, CD;$$

and by Euclid, II. 12, if ACB is obtuse,

$$AB^2 = BC^2 + AC^2 + 2BC \,.\, CD.$$

In first case $CD = b\cos\gamma$, in second case $CD = -b\cos\gamma$.
Hence in both cases

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

39. From the forms in the last Article, we have

$$1-\cos\gamma = 1-\frac{a^2+b^2-c^2}{2ab} = \frac{c^2-(a-b)^2}{2ab},$$

or
$$2\sin^2\frac{\gamma}{2} = \frac{(c-a+b)(c+a-b)}{2ab}.$$

Let $2s = a+b+c$,

and therefore $s-a = \dfrac{b+c-a}{2}$, and so for $s-b$, $s-c$.

And we have
$$\sin^2\frac{\gamma}{2} = \frac{(s-a)(s-b)}{ab}.$$

Again, $1+\cos\gamma = \dfrac{(a+b)^2-c^2}{2ab} = \dfrac{(a+b+c)(a+b-c)}{2ab}$;

$$\therefore\ \cos^2\frac{\gamma}{2} = \frac{s(s-c)}{ab}.$$

So also
$$\tan^2\frac{\gamma}{2} = \frac{(s-a)(s-b)}{s(s-c)}.$$

And
$$\sin^2\gamma = 4\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2} = 4\,\frac{s(s-a)(s-b)(s-c)}{a^2b^2},$$

or
$$\sin\gamma = \frac{2}{ab}\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Similar forms may be proved for the other angles.

From these forms the formulæ of Art. 36 may be directly deduced.

40. *To find the area of a triangle.* The area of a triangle is equal to the rectangle contained by half the base, and the perpendicular upon the base from the opposite angle, that is,

$$\text{area } ABC = \frac{1}{2}BC\,.\,AD = \frac{1}{2}ab\sin\gamma;$$

this gives the area in terms of two sides and the included angle.

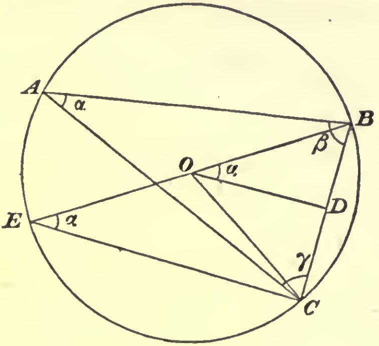
Substituting for $\sin\gamma$ from the preceding Article, we have

$$\text{area } ABC = \sqrt{\{s(s-a)(s-b)(s-c)\}};$$

which is in terms of the sides only.

This expression is frequently denoted by S.

41. *To find the radius of the circumscribed circle.*



Let ABC be a triangle, BE the diameter of the circumscribed circle. Then angle $BEC = BAC = \alpha$; and BCE is a right angle, and we have

$$\frac{BC}{BE} = \sin \alpha, \quad \text{or } 2R = \frac{a}{\sin \alpha};$$

which gives R in terms of one side and the angle opposite to it, R being the radius of the circle.

Substituting for $\sin \alpha$ in terms of the sides we find

$$R = \frac{abc}{4S}.$$

These results may also be obtained from the construction for finding the centre of the circumscribed circle given in Euclid, IV. 5.

For BC being bisected in D, and O being the centre, angle BOC = twice the angle at circumference

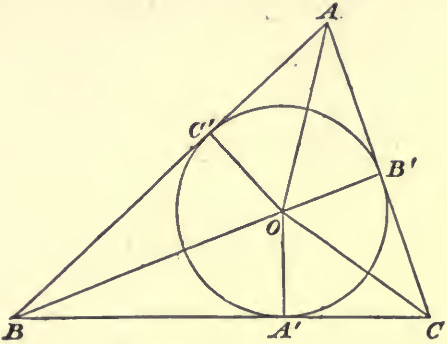
$$= 2\alpha;$$

and therefore $BOD = \alpha, \quad BD = \dfrac{a}{2}$,

and $\dfrac{a}{2} = R \sin \alpha$, as before.

42. *To find the radius of the inscribed circle.*

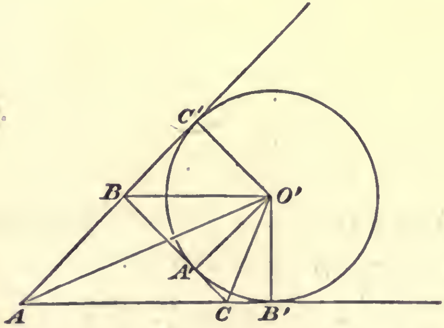
Let O be the centre of the inscribed circle; OA', OB', OC' perpendicular to the sides and equal to r. Then the area of triangle



BOC is $\frac{1}{2}ra$, and so also for triangles BOA, COA. But area of triangle ABC equals the sum of these triangles,

or
$$\frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \text{area of } ABC = S;$$

$$\therefore r = \frac{S}{s}.$$



To find the radii of the escribed circles, that is, of the circles which touch one side, and the other two sides produced. Let O' be centre of the circle touching BC.

Then $$O'A' = O'B' = O'C' = r_a$$

and $$\Delta ABC = \Delta BAO' + \Delta CAO' - \Delta BCO',$$

or $$S = \frac{1}{2} r_a c + \frac{1}{2} r_a b - \frac{1}{2} r_a a,$$

or $$r_a = \frac{S}{s-a};$$

so also $$r_b = \frac{S}{s-b}, \ r_c = \frac{S}{s-c}.$$

EXAMPLES. (A).

Prove the formulæ:

1. $\dfrac{\sin\alpha + \sin\beta}{\sin\beta} = \dfrac{a+b}{b}$.

2. $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \dfrac{a+b}{a-b}$,

3. $\dfrac{\sin\alpha + \sin\beta - \sin\gamma}{\sin\alpha - \sin\beta + \sin\gamma} = \dfrac{a+b-c}{a-b+c}$.

4. $\dfrac{\sin\alpha - \sin\beta}{a-b} = \dfrac{\sin\gamma}{c}$.

5. $\dfrac{a+c-b}{4c} = \dfrac{b\sin^2\frac{\alpha}{2}}{a+b-c}$.

6. $\cos\gamma = \dfrac{\sin^2\alpha + \sin^2\beta - \sin^2\gamma}{2\sin\alpha\sin\beta}$.

7. $a+b+c = (a+b)\cos\gamma + (a+c)\cos\beta + (b+c)\cos\alpha$.

8. $(a+b)(1-\cos\gamma) = c(\cos\alpha + \cos\beta)$.

9. $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \dfrac{\sin\gamma}{1-\cos\gamma} = \cot\dfrac{\gamma}{2}$.

10. $c(1 - \cos^2\beta - \cos^2\alpha) = \cos\gamma\,(a\cos\alpha + b\cos\beta)$.

11. If $a = b$, shew that $\alpha = \beta$.

12. If $c \cos\beta = b \cos\gamma$, shew that the triangle is isosceles $(b=c)$.

13. If $a \sec\beta = 2c$, the triangle is isosceles $(b=c)$.

14. If the triangle is isosceles, $(a=b)$, $c = 2a \sin\frac{\gamma}{2}$.

15. If $c = 2a\sin\frac{\gamma}{2}$, then the triangle is either isosceles, or

$$c = \sqrt{a(a-b)}.$$

16. If $(s-a)(s-b) = ab$, the triangle is impossible.

17. If $s(s-c) = \frac{ab}{2}$, one angle is a right angle.

18. The area of a triangle whose sides are 4, 5, 7 inches respectively, is $4\sqrt{6}$ square inches.

19. Two sides of a triangle are 8 and 10 inches, and the included angle is 30°; the area is 20 square inches.

20. The sides of a triangle are in A.P. and their common difference is 2 inches. If the area is $3\sqrt{15}$ square inches, find the sides. *Ans.* 4, 6, 8 inches.

21. If the area of a triangle is equal to 84 square inches, and two of its sides are 15 and 13 inches, find the third side.

Ans. 14 inches.

22. The sides of a triangle are 3, 7, 8; compare the radii of the inscribed and circumscribed circles. *Ans.* $R : r = 7 : 2$.

23. If the area of a triangle is $\frac{1}{2}ab$, then the angle γ is a right angle.

24. If $R = \frac{c}{2}$, then $\gamma = 90°$.

25. If $r = s - c$, then $a^2 + b^2 = c^2$.

26. Shew that $r_a + r_b = c \cot\frac{\gamma}{2}$.

27. If $r_a = r_b = \frac{1}{2}r_c$, then $c = \frac{4}{3}a$.

28. If $r_a + r_b = r_c$, then $c = \dfrac{a + b \pm 2\sqrt{a^2 - ab + b^2}}{3}$.

29. $b \cos^2 \dfrac{\gamma}{2} + c \cos^2 \dfrac{\beta}{2} = \dfrac{a + b + c}{2}$.

30. $(b - c) \cos \dfrac{\alpha}{2} = a \sin \dfrac{\beta - \gamma}{2}$.

31. $\dfrac{\sin^2 \dfrac{\beta}{2}}{b} + \dfrac{\sin^2 \dfrac{\gamma}{2}}{c} = \dfrac{s - a}{bc}$.

32. $\cot \dfrac{\alpha}{2} + \cot \dfrac{\beta}{2} = \dfrac{c}{r}$.

33. $r = s \tan \dfrac{\alpha}{2} \tan \dfrac{\beta}{2} \tan \dfrac{\gamma}{2}$.

34. $\dfrac{a}{b} \cdot \dfrac{1 - \cos \beta}{1 - \cos \alpha} = \tan \dfrac{\beta}{2} \cot \dfrac{\alpha}{2}$.

35. $\dfrac{\sin \alpha}{\cot \dfrac{\beta}{2} + \cot \dfrac{\gamma}{2}} = 2 \sin \dfrac{\alpha}{2} \sin \dfrac{\beta}{2} \sin \dfrac{\gamma}{2}$.

36. $\dfrac{b - c}{r_a} + \dfrac{c - a}{r_b} + \dfrac{a - b}{r_c} = 0$.

37. $(a + c) \sin \dfrac{\beta}{2} = b \cos \dfrac{\alpha - \gamma}{2}$.

38. If $\sin \alpha$, $\sin \beta$, $\sin \gamma$ are in A.P., then $\tan \dfrac{\alpha}{2} \tan \dfrac{\gamma}{2} = \dfrac{1}{3}$.

39. *ABC* is an isosceles right-angled triangle, one of whose base angles *B* is trisected by the lines *BD*, *BE*. Shew that

$$AD : DE : EC = \sqrt{3} : 2 : \sqrt{3} - 1.$$

40. If *AD* is the perpendicular from *A* on the side *BC* of a triangle *ABC*, and *DE*, *DF* are drawn perpendicular to *AB*, *AC* respectively, shew that $EF \times R =$ area of triangle *ABC*.

EXAMPLES. (E).

Prove the following relations between the sides and angles of a triangle.

1. $a^2+b^2+c^2=2ab\cos\gamma+2ac\cos\beta+2bc\cos\alpha.$

2. $a(b\cos\gamma-c\cos\beta)=b^2-c^2.$

3. $\cos\alpha+\cos\beta=2\dfrac{a+b}{c}\sin^2\dfrac{\gamma}{2}.$

4. $\tan\beta=\dfrac{b\sin\gamma}{a-b\cos\gamma}.$

5. $a^2\sin 2\beta+b^2\sin 2\alpha=2ab\sin\gamma.$

6. $\cot\alpha-\cot\beta=\dfrac{b^2-a^2}{ab\sin\gamma}.$

7. $R=\dfrac{1}{8}(a+b+c)\sec\dfrac{\alpha}{2}\sec\dfrac{\beta}{2}\sec\dfrac{\gamma}{2}.$

8. $\dfrac{1}{a}\cos^2\dfrac{\alpha}{2}+\dfrac{1}{b}\cos^2\dfrac{\beta}{2}+\dfrac{1}{c}\cos^2\dfrac{\gamma}{2}=\dfrac{(a+b+c)^2}{4abc}.$

9. $1-\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{2c}{a+b+c}.$

10. $\dfrac{\operatorname{vers}\alpha}{\operatorname{vers}\beta}=\dfrac{a+c-b}{b+c-a}\cdot\dfrac{a}{b}.$

11. $\dfrac{\cot\dfrac{\beta}{2}+\cot\dfrac{\gamma}{2}}{\cot\dfrac{\alpha}{2}}=\dfrac{2a}{b+c-a}.$

12. $2Rr=\dfrac{abc}{a+b+c}.$

13. $4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}=r.$

14. $r_ar_b+r_ar_c+r_br_c=s^2.$

15. $a \cos \alpha + b \cos \beta + c \cos \gamma = 4R \sin \alpha \sin \beta \sin \gamma.$

16. $\tan^2 \frac{\alpha}{2} = \frac{rr_a}{r_b r_c}.$

17. $r_a r_b r_c = rs^2.$

18. $r_a + r_b + r_c - r = 4R.$

19. $\frac{\sin (\beta - \gamma)}{\sin \alpha} = \frac{b^2 - c^2}{a^2}.$

20. $\cot \frac{\alpha}{4} - \operatorname{cosec} \frac{\alpha}{2} : \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = b + c - a : 2a.$

Prove the following expressions for the area of a triangle.

21. $(rr_a r_b r_c)^{\frac{1}{2}}.$

22. $Rr (\sin \alpha + \sin \beta + \sin \gamma).$

23. $\frac{a^2 - b^2}{2} \frac{\sin \alpha \sin \beta}{\sin (\alpha - \beta)}.$

24. $\frac{2abc}{a + b + c} \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.$

25. $\frac{2s^2 \sin \alpha \sin \beta \sin \gamma}{(\sin \alpha + \sin \beta + \sin \gamma)^2}.$

26. $\frac{a^2}{4} \left\{ \frac{\cos (\beta - \gamma)}{\sin \alpha} + \cot \alpha \right\}.$

27. $s (s - a) \tan \frac{\alpha}{2}.$

28. $\frac{1}{2} \sqrt{a^2 bc \cos \beta \cos \gamma + b^2 ac \cos \alpha \cos \gamma + c^2 ab \cos \alpha \cos \beta}.$

29. $\frac{1}{4} \sqrt[3]{2a^2 b^2 c^2 (\sin 2\alpha + \sin 2\beta + \sin 2\gamma)}.$

30. The areas of all triangles described about the same circle vary as their perimeters.

31. If a^2, b^2, c^2 be in A. P., then

$$\frac{\sin 3\beta}{\sin\beta} = \left(\frac{a^2 - c^2}{2ac}\right)^2.$$

32. If a, b, c be in A. P., then will r_a, r_b, r_c be in H. P.

33. Shew that R, r have a common measure if a, b, c have one.

34. The distances between the centres of the inscribed and the centres of the escribed circles are

$$4R\sin\frac{\alpha}{2},\ 4R\sin\frac{\beta}{2},\ 4R\sin\frac{\gamma}{2},$$

respectively.

35. If p_a, p_b, p_c be the perpendiculars from the angles on the opposite sides, then

$$p_a p_b p_c = \frac{8s^3r^3}{abc}.$$

36. If l be the length of the line bisecting the angle α, then

$$l = \frac{2\sqrt{bcs\,.\,(s-a)}}{b+c}.$$

37. If the perpendiculars from the angles of a triangle ABC on the opposite sides meet in O, then

$$\tan\alpha = \frac{AO}{BC},$$

and $\qquad AO : BO : CO = \sin\alpha\tan\alpha : \sin\beta\tan\beta : \sin\gamma\tan\gamma.$

38. If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are in G. P., then

$$a^4 + c^4 = b^2(a^2 + c^2).$$

39. If a, b, c be the sides of a triangle inscribed in a circle radius R, then

$$R^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2) + a^2b^2c^2 = 0.$$

EXAMPLES. (C).

In any triangle, prove that

1. $\cot\alpha+\cot\beta+\cot\gamma=\dfrac{a^2+b^2+c^2}{4S}$.

2. $b^2\sin 2\gamma-2bc\sin(\beta-\gamma)-c^2\sin 2\beta=0$.

3. $b^2\cos 2\gamma+2bc\cos(\beta-\gamma)+c^2\cos 2\beta=a^2$.

4. $(b^2-c^2)\cot\alpha+(c^2-a^2)\cot\beta+(a^2-b^2)\cot\gamma=0$.

5. $\tan\dfrac{\alpha}{2}+\tan\dfrac{\beta}{2}+\tan\dfrac{\gamma}{2}+\dfrac{s}{r}=4R\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

6. $s=r\left(\cot\dfrac{\alpha}{2}+\cot\dfrac{\beta}{2}+\cot\dfrac{\gamma}{2}\right)=R(\sin\alpha+\sin\beta+\sin\gamma)$.

7. $a^2\tan^2\beta-2ab\tan\beta\sin\gamma+b^2=b^2\cos^2\gamma\sec^2\beta$.

8. $4r^2s=abc(\cos\alpha+\cos\beta+\cos\gamma-1)$.

9. $\dfrac{a}{r_a}-\dfrac{b}{r_b}=\tan\dfrac{\beta}{2}-\tan\dfrac{\alpha}{2}$.

10. $\dfrac{\cos\dfrac{\alpha}{2}\cos\dfrac{\beta-\gamma}{2}}{b+c}=\dfrac{\cos\dfrac{\beta}{2}\cos\dfrac{\alpha-\gamma}{2}}{a+c}=\dfrac{\cos\dfrac{\gamma}{2}\cos\dfrac{\alpha-\beta}{2}}{a+b}$.

11. $\cos^{-1}\sqrt{\dfrac{r_br_c}{bc}}+\cos^{-1}\sqrt{\dfrac{r_ar_c}{ac}}+\cos^{-1}\sqrt{\dfrac{r_ar_b}{ab}}=90°$.

12. $\dfrac{a^2\sin(\beta-\gamma)}{\sin\alpha}+\dfrac{b^2\sin(\gamma-\alpha)}{\sin\beta}+\dfrac{c^2\sin(\alpha-\beta)}{\sin\gamma}=0$.

13. $(a+b+c)(\cos\alpha+\cos\beta+\cos\gamma)$

$$=2a\cos^2\frac{\alpha}{2}+2b\cos^2\frac{\beta}{2}+2c\cos^2\frac{\gamma}{2}.$$

14. $S=s^2\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}$.

15. $S=\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\left\{\frac{a^2}{\sin\alpha}+\frac{b^2}{\sin\beta}+\frac{c^2}{\sin\gamma}\right\}$.

16. The distance of the foot of the perpendicular from the angle A on the side a, from the middle point of a, is $\frac{b^2-c^2}{2a}$.

17. If $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in arithmetical progression, so also are a^2, b^2, c^2.

18. If p_a, p_b, p_c be the perpendiculars from the angles on the opposite sides, then

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}=\frac{1}{r}.$$

19. Shew that $r=\frac{a\sin\beta\sin\gamma}{\sin\alpha+\sin\beta+\sin\gamma}=\frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$.

20. Shew that $r_a=\frac{a\sin\beta\sin\gamma}{\sin\beta+\sin\gamma-\sin\alpha}=\frac{a\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$.

21. Shew that $r_a+r_b+r_c=2R\left(\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}\right)$.

22. If γ be a right angle,

$$\cos(2\alpha-\beta)=\frac{a}{c^3}(3c^2-4a^2).$$

23. The perpendicular from C on the opposite side AB is

$$\frac{ab}{2}\cdot\frac{a\sin\alpha+b\sin\beta+c\sin\gamma}{ab\cos\gamma+ac\cos\beta+bc\cos\alpha}.$$

24. If $\alpha:\beta:\gamma=2:3:4$, then $\frac{a+c}{2b}=\cos\frac{\alpha}{2}$.

25. The length of the line drawn bisecting the angle α, and meeting the side a is $\frac{2bc}{b+c}\cos\frac{\alpha}{2}$.

26. In any triangle the perpendicular from the angle A on the side BC is equal to $\dfrac{b^2 \sin\gamma \pm c^2 \sin\beta}{b \pm c}$.

27. If $A'B'C'$ be the triangle formed by joining the feet of the perpendiculars from the angles A, B, C on the opposite sides, then $B'C' = R \sin 2\alpha$.

28. If $a'b'c'$ be the sides of the triangle $A'B'C'$, then

$$\frac{a'}{a^2} + \frac{b'}{b^2} + \frac{c'}{c^2} = \frac{a^2 + b^2 + c^2}{2abc},$$

and
$$a' + b' + c' = \frac{abc}{2R^2}.$$

29. If a^2, b^2, c^2 be in A.P., then $a \sec\alpha$, $b \sec\beta$, $c \sec\gamma$ are in H.P.

30. If $\tan\dfrac{\alpha}{2} \tan\dfrac{\beta}{2} = \dfrac{n-1}{n+1}$, then

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \frac{\sin\alpha}{n - \cos\alpha} + \frac{2}{n+1} \cdot \frac{n - \cos\alpha}{\sin\alpha}.$$

31. The distance between the centres of the escribed circles touching the sides a, b is

$$\frac{2c\sqrt{ab}}{\sqrt{c^2 - (a-b)^2}}.$$

32. If s, s' be the semi-perimeters of two triangles, and $r_a = r'_a$; then $\dfrac{s-a}{s} : \dfrac{s'-a'}{s'}$ is the ratio of the radii of the inscribed circles.

33. Prove that $\dfrac{\sin\beta}{\sin(\alpha - \beta)} = \dfrac{b}{a\cos\beta - b\cos\alpha}$.

34. If p, q, r be the perpendiculars from the angles upon the opposite sides a, b, c, respectively; then $\dfrac{p^2}{qr} = \dfrac{bc}{a^2}$.

35. The distance of the points of contact of the inscribed circle with the sides AB, AC from the angle A are $s - a$.

36. If a, b, c be in arithmetical progression, then

$$\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}=\frac{1}{3}.$$

37. If the sides of a triangle are in arithmetical progression, then the cotangents of the half angles are in arithmetical progression.

38. In a right-angled triangle (γ the right angle),

$$\sin\frac{\beta}{2}=\sqrt{\left(\frac{c-a}{2c}\right)}.$$

39. If O be the centre of the circles inscribed in a triangle ABC, and R_a, R_b, R_c the radii of the circles circumscribed about the triangles BOC, AOC, AOB, respectively, then

$$\frac{R_a}{\sin\frac{\alpha}{2}}=\frac{R_b}{\sin\frac{\beta}{2}}=\frac{R_c}{\sin\frac{\gamma}{2}}.$$

40. Shew also that the distance of the chords common to these circles and the inscribed circle are distant from O,

$$\frac{r^2}{2R_a},\ \frac{r^2}{2R_b},\ \frac{r^2}{2R_c},$$

respectively.

41. If O be the centre and D, E, F the points of contact of the circle inscribed in a triangle ABC, then

$$OA\,.\,OB\,.\,OC\,(AF+BD+CE)=4R\,.\,AF\,.\,BD\,.\,CE.$$

42. If O be the centre of the inscribed circle of a triangle ABC, and O_a, O_b, O_c the centres of the escribed circles, shew that

$$\frac{AO}{AO_a}+\frac{BO}{BO_b}+\frac{CO}{CO_c}=1.$$

43. If R_a, R_b, R_c be the radii of the circles circumscribed about AEF, BDF, CDE, (Ex. 41) respectively, then

$$aR_a^2+bR_b^2+cR_c^2=\frac{abc}{4}.$$

44. If AD be drawn through O the centre of the inscribed circle then the radii of the circles circumscribing ADB, ADC are together equal to $2R\cos\frac{\alpha}{2}$.

45. If ρ_a, ρ_c be the radii of circles passing through A and touching the side BC of the triangle ABC in the points B, C, then $\rho_a \rho_c = R^2$, and the radius of circle touching BC in its middle point is

$$\frac{2(b^2+c^2)-a^2}{8b \sin\gamma}.$$

46. If $A'B'C'$ be points in the sides BC, AC, AB of a triangle such that

$$BA' = \frac{1}{3}a,\ CB' = \frac{1}{3}b,\ AC' = \frac{1}{3}c;$$

then the area of the triangle $A'B'C'$ is $\frac{S}{3}$.

47. An isosceles triangle is described about two circles (radii r, r') which touch each other, so that its equal sides touch both the circles, and its base the larger circle; shew that its base is $2r\sqrt{\frac{r}{r'}}$, and its sides are $r \cdot \frac{r+r'}{r-r'} \cdot \sqrt{\frac{r}{r'}}$.

48. If O be the centre of the circumscribed circle, and r_1, r_2, r_3 the radii of the circles circumscribed about COB, AOC, BOA respectively, then

$$r_1 \cos\alpha = r_2 \cos\beta = r_3 \cos\gamma.$$

49. If D be any point in the sides BC of a triangle ABC, and if k be the harmonic mean between the radii of the circles described about the triangles ADB, ADC, then

$$\frac{AD}{k} = \sin\beta + \sin\gamma.$$

50. ABC is an isosceles triangle ($AB = AC$): on BC as diameter a circle is described meeting AB, AC in D, E; DC, EB intersect in O. Shew that area $ABOC = \frac{1}{2}a^2 \cot\alpha$.

51. A tangent drawn from a point outside a circle (radius r) makes an angle 2θ with the diameter through that point. Shew that the distance of the point from the circle is

$$\frac{r(1-\tan\theta)^2}{2\tan\theta}.$$

52. If sides BA, CA of a triangle be produced to D, E so that $AD = AE$, and the area of the triangle ADE equals the area of the triangle ABC: shew that the radius of the circle inscribed in DAE is equal to

$$\frac{S}{\sqrt{bc}\left(1+\sin\frac{a}{2}\right)}.$$

53. If lines be drawn bisecting the angles of a parallelogram, the area of the rectangle formed by these lines equals $\frac{1}{2}(a-b)^2\sin\alpha$, where a, b are the sides of the parallelogram, and α the angle between them.

54. If a triangle have sides, which vary inversely as the Harmonic means of the radii of the escribed circles taken two and two, it is similar to the original triangle.

55. Find the length of the line bisecting the triangle ABC and making equal angles with the sides AB, AC, and find the condition that the problem is possible.

Ans. length $= \sqrt{2\,.\,s-b\,.\,s-c}$, and one side must not be greater than twice the other.

56. Three circles (radii r_1, r_2, r_3) touch one another: shew that the radius of the circle passing through the three points of contact is

$$\sqrt{\frac{r_1r_2r_3}{r_1+r_2+r_3}}.$$

Also if a, b, c be the chords joining the points of contact of the three circles, then

$$\frac{8}{abc}=\left(\frac{1}{r}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right).$$

SECTION V.

SOLUTION OF TRIANGLES. FOUR CASES. THEIR TREATMENT IN EUCLID. APPLICATION.

43. EVERY triangle has six parts, three angles and three sides; these parts are not independent of each other, for the three angles are always equal to two right angles, that is, we always have

$$\alpha+\beta+\gamma=180^\circ \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{I.}$$

Also between the sides and angles two other relations independent of each other can be proved to exist. Take for instance the relations proved in Art. 36,

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b} \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{II.}$$

$$\frac{\sin\alpha}{a}=\frac{\sin\gamma}{c} \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{III.}$$

From I. II. III. all other relations between the sides and angles may be deduced. If we had taken any other two of the relations proved in the last section which are independent of each other, we could with equation I. have deduced all the rest.

Hence since we have three and only three independent equations amongst the six parts of a triangle, if three parts are given, we are enabled to determine the other three. Also three parts at least must be given; but if more were, we should find one or more equations between them to express the condition that these parts may belong to the same triangle.

We may observe also, that the three given parts must not all be angles; for by equation I. if we know two angles we also know the other, and so having three angles given would really amount to knowing only two parts; one part then at least must be a side. Also we must not have two sides given us less than the third side, nor such as to make the sines and cosines involved in our formulæ greater than unity. (See Art. 46.)

The process of finding the other parts of a triangle, when three of them are known, is called solving the triangle.

44. In solving triangles we shall find four distinct cases present themselves.

Case I. When the three sides are given.

Case. II. When two sides and an angle opposite to one of them are given.

Case III. When two sides and the included angle are given.

Case IV. When two angles and a side are given.

We shall treat each of these in order.

45. Case I. Let a, b, c be given; to find α, β, γ, we have, Art. 38,

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab};$$

these equations completely determine the angles α, β, γ, but are not used because they are not adapted for logarithmic computation, as will be shewn hereafter (Art. 61). In practice we use the forms derived from them in Art. 39.

46. Case II. Let a, b, α be known: to find c, β, γ:

we have $$\sin\beta = \frac{b}{a}\sin\alpha,$$

from which we find β: we then know

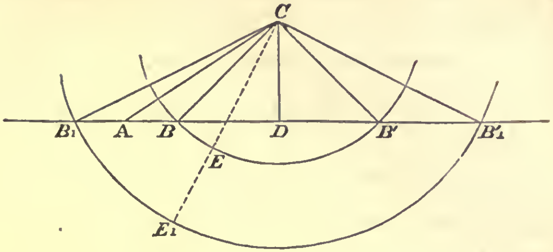
$$\gamma = 180 - (\alpha + \beta),$$

which gives γ: and we then know c from the equation

$$c = a\frac{\sin\gamma}{\sin\alpha}.$$

This case presents a peculiarity which we will now point out.

Let the angle $BAC = \alpha$; $AC = b$; CE, any line drawn from C, equal to a. With centre C and distance CE describe a circle cutting AB in B, B': join CB, CB'; draw CD perpendicular to AB.



If $CE < CD$, the circle will not cut AB at all, and there will be no triangle with the parts proposed, that is, there is no solution.

If $CE > CD$ and $< CA$, two triangles will be formed, CAB, CAB', each having the angle $CAB = \alpha$, the sides $CA = b$ and CB or CB' equal to a: that is to say, there are two solutions of the triangle.

If $CE > CA$, then the circle will meet AB in points B_1, B_1'. Now CAB_1 is not a solution, inasmuch as the angle CAB or α is not a part of it, but CAB_1 or $180° - \alpha$ instead. Hence in this case there is only one solution CAB_1'.

Now let us examine how these cases are indicated by the Trigonometrical formulæ. We have $\sin\beta = \frac{b \sin\alpha}{a}$.

Now $CD = b\sin\alpha$, $CE = a$, and if $CE < CD$, $\sin\beta$ is greater than unity, which is impossible, and therefore there can be no solution in this case.

If $CE > CD$ and $< CA$, then $\sin\beta$ is possible, and $b > a$ and therefore $\beta > \alpha$. Now two values β and $180 - \beta$ will satisfy the equation

$$\sin\beta = \frac{b}{a}\sin\alpha,$$

and since β may be either greater or less than 90° both these values will give solutions of the triangle.

If $CE > CA$, $b < a$, and therefore $\beta < \alpha$ and β must be less than 90°; and only the smaller of the values β and $180 - \beta$ will give a solution of the triangle.

From the fact that when $b > a$ there are two solutions of the triangle, this case is called the ambiguous case.

When α is obtuse, β must be acute, and $a > b$, and there is no ambiguity.

47. Case III. Let a, b, γ be known: to find c, α, β we have

$$c^2 = a^2 + b^2 - 2ab\cos\gamma;$$

this gives the side c: we can find α and β from the formulæ

$$\sin\alpha = \frac{a}{c}\sin\gamma,$$

$$\sin\beta = \frac{b}{c}\sin\gamma.$$

In determining α, β in this manner we should have to consider the order of magnitude of α, β, γ from knowing that of a, b, c; and so determine whether α or $180 - \alpha$, β or $180 - \beta$ are the true solutions. No ambiguity will really present itself.

The following method will avoid this difficulty as well as be better adapted for logarithms, as will be shewn hereafter.

Since
$$\frac{\sin\alpha}{\sin\beta} = \frac{a}{b},$$

we have
$$\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{a-b}{a+b},$$

or
$$\frac{2\sin\dfrac{\alpha-\beta}{2}\cos\dfrac{\alpha+\beta}{2}}{2\cos\dfrac{\alpha-\beta}{2}\sin\dfrac{\alpha+\beta}{2}} = \frac{a-b}{a+b},$$

or
$$\tan\frac{\alpha-\beta}{2} = \frac{a-b}{a+b}\tan\frac{\alpha+\beta}{2}.$$

Now
$$\frac{\alpha+\beta}{2} = 90^\circ - \frac{\gamma}{2};$$

$$\therefore \tan\frac{\alpha-\beta}{2} = \frac{a-b}{a+b}\cot\frac{\gamma}{2}.$$

This equation will give $\frac{\alpha-\beta}{2}$, and since $\alpha+\beta$ is known we can find α and β.

The side c will then be found from the equation

$$c = a\frac{\sin\gamma}{\sin\alpha}.$$

48. Case IV. First, let a, β, γ be given; to find b, c, α. We have

$$\alpha = 180^\circ - (\beta+\gamma),$$

which gives α, and then b, c are determined from the equations

$$b = a\frac{\sin\beta}{\sin\alpha},$$

$$c = a\frac{\sin\gamma}{\sin\alpha}.$$

Secondly, let a, α, β be given; to find b, c, γ. We have

$$\gamma = 180^\circ - (\alpha+\beta),$$

and then b, c are found as before.

49. It will perhaps be of advantage to the student to compare these cases with the corresponding propositions in Euclid.

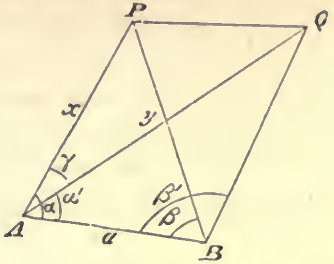
Euclid (I. 8) proves that if two triangles have their sides equal, they have their angles equal; in other words, that if three sides of a triangle are fixed, then the angles cannot vary; this agrees with Case I.

Euclid (I. 4) in the same way agrees with Case III., and (I. 26) with the two parts of Case IV.

Case II. we have already considered geometrically, but the student would do well to compare it with Euclid, VI. 7.

50. We give the following as an example of the application of the rules in this section to practical purposes.

Let it be required to find the distance between the summits P, Q of two inaccessible mountains. At A let the observer measure the angles $PAB=\alpha$, $QAB=\alpha'$, $PAQ=\gamma$; at B let him



measure $PBA = \beta$, $QBA = \beta'$, and also let him measure the distance $AB = a$. Let $AP = x$, $AQ = y$. Then he would proceed with his calculations as follows:

$$AP = x = a\frac{\sin\beta}{\sin(\alpha+\beta)},\quad y = a\frac{\sin\beta'}{\sin(\alpha'+\beta')},$$

which give x and y. PQ is then known from the equation

$$PQ^2 = x^2 + y^2 - 2xy\cos\gamma.$$

EXAMPLES. (A).

1. If $a = 2$, $b = 3$, $c = 4$; solve the triangle.
   Ans. $\alpha = 28°.\ 57'$; $\beta = 46°.\ 34'$; $\gamma = 104°.\ 29'$.

2. If $a = 4$, $b = 5$, $c = 6$; solve the triangle.
   Ans. $\alpha = 41°.\ 36'$; $\beta = 55°.\ 46'$; $\gamma = 82°.\ 38'$.

3. If $a = 3$, $b = 7$, $c = 5$; solve the triangle.
   Ans. $\alpha = 21°.\ 47'$, $\beta = 120°$, $\gamma = 38°.\ 13'$.

4. If $a = 5$, $b = 6$, $c = 7$; solve the triangle.
   Ans. $\alpha = 44°.\ 25'$, $\beta = 57°.\ 7'$, $\gamma = 78°.\ 28'$.

5. If $a = 11$, $b = 15$, $\gamma = 78°.\ 28'$; solve the triangle.
   Ans. $c = 16{\cdot}7332$ nearly, $\alpha = 40°.\ 6'$, $\beta = 61°.\ 26'$.

6. If $a = 9$, $b = 10$, $\gamma = 60°$, find c. *Ans.* $c = 9{\cdot}539$.

7. If $a = 21$, $b = 15\sqrt{2}$, $\gamma = 45°$; solve the triangle.
   Ans. $c = 16{\cdot}1554$, $\alpha = 66°.\ 48'$, $\beta = 68°.\ 12'$.

8. If $\alpha = 68°.\ 13'$, $\beta = 45°.\ 35'$, $a = 25$; find b.
   Ans. $b = 19{\cdot}2$ nearly.

9. If $\alpha = 60°$, $b = 20$, $c = 10$; solve the triangle.

Ans. $\beta = 90°$, $\gamma = 30°$, $a = 17{\cdot}32$.

10. If $\alpha = 38°.\,13'$, $\beta = 21°.\,47'$, $c = 77$; solve the triangle.

Ans. $a = 55$, $b = 33$, $\gamma = 120°$.

11. If $\beta = 30°$, $\gamma = 45°$, $a = 7$; solve the triangle.

Ans. $\alpha = 105°$, $b = 3{\cdot}62$, $c = 5{\cdot}12$.

12. If $b = d\sqrt{2}$, $c = (\sqrt{3} + 1)\,d$, $\alpha = 45$; solve the triangle.

Ans. $a = 2d$, $\beta = 30°$, $\gamma = 105°$.

13. Having given the lines p, q, r drawn from the angles A, B, C of a triangle to bisect the opposite sides, shew that

$$a = \frac{2}{3}\sqrt{2\,(q^2 + r^2) - p^2}.$$

If γ is a right angle, shew that $p^2 + q^2 = 5r^2$.

14. A lake is bounded by a vertical cliff whose height is equal to the breadth of the lake (a): from a balloon above the lake they subtend the same angle α. Shew that the height of the balloon is

$$\frac{a}{2\sin\alpha}(\sin\alpha + \cos\alpha).$$

15. The courses of two ships are N., and E., and their rates of sailing are as $1 : \sqrt{2}$. The bearing of the former to the latter was E.N.E., but after the latter had sailed 4 miles, it was N.W. What was the original distance of the ships?

Ans. $2\sqrt{20 - 14\sqrt{2}}$ miles.

16. If in a triangle ABC, AD, BE, CF are drawn perpendicular to the sides from the opposite angles, and a new triangle is formed whose sides are AD, BE, CF, shew that the cosine of the angle opposite to AD is $\dfrac{a^2b^2 + a^2c^2 - b^2c^2}{2a^2bc}$ where a, b, c are the sides of ABC.

17. If in a triangle, whose sides are a, b, c, the lines drawn from the angles to bisect the opposite sides are p_1, p_2, p_3, then

$$p_1^2 + p_2^2 + p_3^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

EXAMPLES. (B).

1. If $a = 3$, $b = 5$, $\alpha = 21^\circ.\ 47'$; find c.

Ans. $c = 7$, or $2{\cdot}28$ nearly.

2. If $a = 5$, $b = 6$, $\alpha = 32^\circ.\ 53'$; solve the triangle.

Ans. $\beta = 40^\circ.\ 39'$ or $139^\circ.\ 21'$; $\gamma = 106^\circ.\ 28'$ or $7^\circ.\ 46'$;
$c = 8{\cdot}8$ or $1{\cdot}245$.

3. In the ambiguous case, if α, a, b be given, shew that the two values γ, γ' of the third angle may be found from the equations

$$\gamma + \gamma' = 180^\circ - 2\alpha,\ \cos\frac{\gamma - \gamma'}{2} = \frac{b}{a}\sin\alpha.$$

4. If c, c' be the two values of the third side in the last question, then $c + c' = 2b\cos\alpha$, $c \sim c' = 2a\cos\beta$.

Shew also that $(c - c')^2 + (c + c')^2\tan^2\alpha = 4a^2$.

Also $cc' = b^2 - a^2$.

5. In the ambiguous case if $a + b = 2c$, shew that $a \sim b = \frac{c'}{2}$.

6. In the ambiguous case if $\sin\alpha = \frac{a}{2b}$, shew that the angle between the two positions of a is 120°.

7. In the ambiguous case if k, k be the areas of the two triangles formed, shew that

$$k + k' = \frac{b^2}{2}\sin 2\alpha, \qquad kk' = \frac{1}{4}b^2(b^2 - a^2)\sin^2\alpha.$$

8. The sides of a triangle are in A. P. and its area : the area of an equilateral triangle of the same perimeter = 3 : 5 ; shew that its sides are as 3 : 5 : 7 and one of its angles is 120°.

9. If $\cos\gamma = \frac{\sin\alpha}{2\sin\beta}$, and $\sin^2\alpha = \sin^2\beta + \sin^2\gamma$, find the angles.

Ans. $\alpha = 90^\circ$, $\beta = \gamma = 45^\circ$.

10. If AD be drawn bisecting the angle A of a triangle, and β, $ADC = \delta$, $DC = d$, be known, solve the triangle.

11. If a triangle be formed whose sides are $a+b$, $a+c$, $b+c$ respectively, and γ' is the angle opposite to $a+b$, then

$$\sin^2 \frac{\gamma'}{2} = \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \operatorname{cosec}\left(\alpha + \frac{\beta}{2}\right) \operatorname{cosec}\left(\beta + \frac{\alpha}{2}\right).$$

12. If α, a, $b+c$ are known, shew that β and γ may be found from the equation

$$\cos \frac{\beta - \gamma}{2} = \frac{b+c}{a} \sin \frac{\alpha}{2}.$$

13. Given α, β, and $a+b$; solve the triangle.

14. Given the perimeter $2s$, the area S, and the angle α of a triangle, find the side a.

Ans. $a = s - \frac{S}{s} \cot \frac{\alpha}{2}$.

15. The angles of a triangle are as 3, 4, 5, and the least side is a; find all the sides.

Ans. $\frac{a}{2}\sqrt{6}$, $\frac{a}{2}(\sqrt{3}+1)$.

16. Given the vertical angle, the base, and the difference of the two sides of the triangle; find the other angles.

17. A straight line of length p bisects the angle BAC, and divides the side BC into parts m, n. Shew that

$$p^2 = bc - mn.$$

18. If perpendiculars be drawn from the angles of a triangle upon the opposite sides, and the feet of these perpendiculars be joined to form a new triangle, its sides are $a \cos \alpha$, $b \cos \beta$, $c \cos \gamma$ respectively.

19. Find the height of a cloud by observing its elevation α, and its depression β when seen by reflection in a lake from a station at a height h above the surface of the lake

Ans. Height above the lake is $h \frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$.

20. AB, CD are two towers in the same horizontal plane. The height of AB is h, the elevation of D at A is α, at B is β.

Shew that the height of CD is $\dfrac{h \sin \alpha \cos \beta}{\sin (\alpha - \beta)}$;

and the distance AC is $\dfrac{h \cos \alpha \cos \beta}{\sin (\alpha - \beta)}$.

21. Two ships are lying at anchor at a known distance from each other: find by observations made on board the two ships the distance from either of them of an object on the shore.

22. Find c in the equation

$$b^2 + c^2 - 2bc \cos \alpha = a^2,$$

and shew that the ambiguous case is involved in the result.

23. From the top A of a tower AB (height h) the angles of depression of two objects C, D, in the horizontal plane with the foot of the tower are observed to be $45° - \alpha$, $45° + \alpha$ respectively, and the angle subtended by CD is 2α: shew that $CD = 2h \tan 2\alpha$, and $\angle ACD = 45° - \alpha$.

EXAMPLES. (C.)

1. In any triangle $\tan\left(\dfrac{\alpha}{2} + \beta\right) = \dfrac{c+b}{c-b} \tan \dfrac{\alpha}{2}$.

2. A circle of radius r is inscribed in a sector of a circle of radius a, the chord of the sector being equal to $2c$; shew that

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{c}.$$

3. Two inaccessible objects C, D are observed from two stations A, B in the same plane with them, so that

$$CAD = \alpha = CBD,\ BAD = \beta,\ ABC = \gamma,\ AB = c:$$

shew that $CD = \dfrac{c \sin \alpha}{\sin (\alpha + \beta + \gamma)}$.

4. In the ambiguous case if c, c' be the ambiguous sides and θ the angle between the two positions of a, then

$$\tan\theta = \frac{(c^2 - c'^2)\sin 2\alpha}{2cc' - (c^2 + c'^2)\cos 2\alpha}.$$

5. In the ambiguous case (a, b, α given), if one triangle be n times the other, then

$$\frac{a}{b} = \frac{1}{n+1}\sqrt{n^2 + 1 - 2n\cos 2\alpha}.$$

6. A, B, C, D are four trees in a straight row, such that AB, BC, CD subtend equal angles at a point P. If $AB = 40$ ft., $BC = 20$ ft., $CD = 60$ ft., find PA, PB, PC, PD.

Ans. $PA = 24\sqrt{5}$ ft. $PB = 8\sqrt{10}$ ft. $PC = 12\sqrt{5}$ ft. $PD = 24\sqrt{10}$ ft.

7. In what direction must a road be carried up a hill, whose inclination to the horizon is α, in order that the ascent may be 1 in n? *Ans.* An angle $\operatorname{cosec}^{-1}(n\sin\alpha)$ with the base of the hill.

8. Two towers (heights a, b) stand on a horizontal plane at a distance c from each other: if θ be the angle subtended by the tower (a) at the top of the other, then $\cot\theta = \dfrac{b^2 + c^2 - ab}{ac}$.

9. A tower AB stands on an inclined plane; at a point C on the plane the tower is observed to subtend an angle α; and on proceeding to a point D in the line AC, so that $CD = CA$, it subtends an angle β. If ϕ be the angle between the tower and the line AC, then $\cot\phi = \cot\beta - 2\cot\alpha$.

10. A person in the valley between two hills observes the altitude α of one of them. He then ascends the other through a vertical height a, and finds the angle of depression of his former station β, and the altitude of the hill $(\alpha - \beta)$. The height of the observed hill is $a\dfrac{\sin^2\alpha}{\sin^2\beta}$.

11. A tower situated on a horizontal plane leans towards the north: at two points due south and distant a, b respectively from the foot of the tower, the altitudes of the tower are observed to be

α, β respectively: the vertical height of the tower is $\dfrac{b-a}{\cot\beta - \cot\alpha}$, and its inclination to the horizon is $\tan^{-1}\dfrac{b-a}{b\cot\alpha - a\cot\beta}$.

12. A quadrilateral figure is inscribed in a semicircle (diameter a), and the sides drawn from the extremities of the diameter are inclined to each other at an angle α. Shew that the side of the quadrilateral opposite to the diameter is equal to $a\cos\alpha$.

13. A person at A observes the elevation α of the summit C of a mountain: he then proceeds up a slope (inclination γ) directly towards the summit to B; at B the elevation of the mountain is β: if $AB = a$, shew that the height of the mountain is

$$a \sin\alpha \sin(\beta-\gamma) \operatorname{cosec}(\beta-\alpha).$$

14. A person observes two telegraph posts in the same line, which is at right angles to the road on which he is walking. After walking a yards the nearer post is in a straight line with a church; after b more yards the further post and church are in the same line, and the two posts subtend the same angle as they did at his first station. If the distance of the nearer post from the first station is l, shew that the distance of the church is $\dfrac{lab}{l^2 - 2a^2}$.

SECTION VI.

LOGARITHMS. THEIR USE. ADVANTAGE OF LOGARITHMS TO BASE 10. ARRANGEMENT OF THE TABLES. LOGARITHMS OF THE TRIGONOMETRICAL RATIOS.

51. We have established in the preceding sections certain relations between the sides and angles of triangles by means of which when we know certain parts of a triangle, we are enabled to calculate the rest: and our examples have illustrated the way in which these relations may be applied to calculate the heights and distances of objects which cannot be directly measured. In most practical cases these methods would involve tedious numerical calculations, and it becomes a matter of great importance to shorten them. This is done by means of what are called logarithms, as we shall now proceed to explain.

52. In the equation $N = a^x$, where N, a, x are any quantities satisfying this relation, a is called the *base*, and x the *logarithm* of the number N to the base a. This relation is also thus expressed, $x = \log_a N$.

Hence we obtain the following definition. The logarithm of a number to a given base, is the index of the power to which the base must be raised in order to be equal to the given number.

53. In the equation $N = a^x$, let a be some constant quantity greater than unity: then as x gradually increases, N gradually increases;

$$\text{when } x = 0,\ N = 1; \quad \text{when } x = 1,\ N = a;$$
$$\text{when } x = 2,\ N = a^2; \quad \text{and when } x = \infty,\ N = \infty.$$

As x increases from 0 to 1, N increases from 1 to a; and whatever value x may have between 0 and 1, there must be some corresponding value of N between 1 and a. And generally, whatever positive value be given to x, there must be some corresponding value of N between 1 and infinity. And if we give x all possible values between 0 and ∞, the corresponding values of N will comprise all possible values between 1 and ∞.

Again, if x receive all possible values from 0 to $-\infty$, N must receive all fractional values between $\frac{1}{a^0}$ and $\frac{1}{a^\infty}$, that is, between 1 and 0.

Hence, by giving x all values from $-\infty$ to $+\infty$, N will have all values from 0 to ∞, that is, all positive values.

And conversely, whatever positive value N may have, there must be some real value of x, such that $a^x = N$.

If a be less than unity, let $a = \frac{1}{b}$, then $a^x = b^{-x} = N$; and by giving x all values from $+\infty$ to $-\infty$, we give b^{-x} all values from 0 to ∞, and therefore a^x or N all values from 0 to ∞.

If $a = 1$, all real powers of a are equal to unity, and we cannot make $a^x = N$.

If a is negative, the values of a^x are sometimes positive, sometimes negative, sometimes impossible, and so we cannot always make it equal to N. So also if N is negative we cannot always find x, such that $a^x = N$.

Hence we conclude, that if a is any positive quantity other than unity, we can always find a quantity N equal to a^x whatever real quantity x may be, and we can always find a value of x, such that $a^x = N$, whatever positive quantity N may be.

Now in order to make logarithms of any practical advantage, we must have the logarithms of all numbers up to a certain magnitude, calculated to a given base, arranged in tables; such an arrangement is called a table of logarithms, and the series of numbers and their corresponding logarithms is called a system of logarithms. Hence we see that the base of our system of logarithms must be positive, and the logarithms of negative numbers must be excluded.

We shall assume in the present treatise that the logarithms of all numbers to any positive base other than unity can be calculated to any degree of accuracy, and arranged in tables. The use of such tables is all that falls within our province.

54. Let x, y be the logarithms of the same number N to the bases a, b respectively. Then

$$x = \log_a N, \quad y = \log_b N,$$

and we have

$$a^x = N = b^y;$$

$$\therefore\ a^{\frac{x}{y}} = b;$$

$$\therefore\ \frac{x}{y} = \log_a b;$$

$$\therefore\ \log_b N = \frac{1}{\log_a b} \cdot \log_a N.$$

Hence if we know the logarithm of any number N to base a, and also the logarithm of b to the same base a, we can find the logarithm of N to the base b, by multiplying $\log_a N$ by the fraction $\frac{1}{\log_a b}$. This multiplier is called the *modulus* of the system whose base is b to the system whose base is a, because by means of it we can reduce a system of logarithms with base a, to a system with base b.

To prove that $\log_a b \times \log_b a = 1$.

Let $$x = \log_a b, \quad y = \log_b a,$$

then $$a^x = b, \quad b^y = a,$$

$$\therefore\ a^{xy} = (a^x)^y = b^y = a,$$

$$\therefore\ xy = 1;$$

or $$\log_a b \times \log_b a = 1,$$

which was to be proved.

55. To find the logarithm of a product or quotient; of a power or root.

Let $$x = \log_a M, \quad y = \log_a N,$$

then $$M = a^x, \quad N = a^y,$$

$$\therefore MN = a^x \cdot a^y = a^{x+y},$$

$$\therefore\ \log_a (MN) = x + y = \log_a M + \log_a N;$$

similarly $$\log_a (MNP \ldots) = \log_a M + \log_a N + \log_a P + \ldots$$

Again, $$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y},$$

$$\therefore \log_a \left(\frac{M}{N}\right) = x - y = \log_a M - \log_a N.$$

Again, since $$M = a^x,$$

$$\therefore M^r = a^{rx},$$

$$\therefore \log_a M^r = rx = r \log_a M.$$

This is true whether r be whole or fractional,

$$\therefore \log_a \sqrt[r]{M} = \log_a M^{\frac{1}{r}} = \frac{1}{r} \log_a M.$$

It is on these properties that the great utility of logarithms depends—for the logarithm of the product of two numbers, we here see, is equal to the sum of the logarithms of the numbers, and so a process of multiplication is reduced to one of simple addition. And in the same way we facilitate the operations of division, involution, and evolution, by reducing them to subtraction, multiplication, and division respectively.

Ex. To find the value of $\sqrt[16]{7985683}$.

We find from the tables that

$$\log_{10} 7985683 = 6{\cdot}9023122,$$

$$\therefore \log_{10} \sqrt[16]{7985683} = \frac{6{\cdot}9023122}{16} = {\cdot}4313945\,;$$

but from the tables we have

$${\cdot}4313945 = \log_{10} 2{\cdot}700191,$$

$$\therefore \sqrt[16]{7985683} = 2{\cdot}700191.$$

56. Let the logarithm of any number N to base 10 be $n + d$, where n is some integer, positive, or negative, or zero; and d a decimal fraction; then

$$\log_{10} N = n + d,$$

$$\text{and } \log_{10} 10^m N = \log_{10} 10^m + \log_{10} N = m + n + d.$$

Hence we see that the logarithms of N and $10^m N$ to base 10 have the same decimal part, whatever integer m is, positive or negative. And if we have a system of logarithms calculated to base 10, the same decimal part of the logarithm will correspond to all numbers which differ from each other only in the position of the decimal point, or which can be derived from each other by multiplying or dividing by any power of 10. Numbers of this kind are said to have the same *significant* digits. It is from this property that 10 is chosen as the base of the system of logarithms in common use; and for the future we shall speak of the logarithm of a number to the base 10, as simply the logarithm of the number, and we shall leave out the suffix in writing it; thus by $\log N$ we shall mean $\log_{10} N$.

57. The integral part of a logarithm is called its *characteristic*, the decimal part its *mantissa*. We have shewn that all numbers having the same significant digits have the same mantissæ, we now proceed to shew how the characteristic may be determined in any particular case by simple inspection.

Let N be a number which has one digit before the decimal point, that is to say, some number between 1 and 10: then $\log N$ must lie between 0 and 1, and is therefore some decimal.

Let $$\log N = d,$$

then $10^{n-1} N$ is a number which has n figures before the decimal point, and its logarithm is $(n-1)+d$. Hence we see that if a number has n figures before the decimal point, its characteristic is $(n-1)$.

Again, $\frac{N}{10^{n+1}}$ has n cyphers after the decimal point, before the first significant digit, and its logarithm is $-(n+1)+d$. Hence, if there are n cyphers after the decimal point, the characteristic is $-(n+1)$.

This last case requires some explanation:

$$-(n+1)+d = -n-(1-d) = -n-d' = -(n+d'),$$

where $d' = 1 - d$ and is some decimal. Here the logarithm is negative, and the integral part $-n$: also the mantissa d' of $\log \frac{N}{10^{n+1}}$ is not the same as d the mantissa of $\log N$, which seems to contradict what was before observed: but it is found more convenient to consider all numbers as having logarithms with positive mantissæ, and it is with this convention that we say that $\log N$ and $\log \frac{N}{10^{n+1}}$ have the same mantissæ, and that the characteristic of the latter is $-(n+1)$. Without this convention we should be obliged to have two tables, one for numbers greater, and the other for numbers less than unity.

Ex. $\log 5{\cdot}4952 = {\cdot}7399835;$

$\therefore \log 5495{\cdot}2 = 3{\cdot}7399835,$

and $\log {\cdot}0054952 = \bar{3}{\cdot}7399835,$

the $-$ sign placed above the 3 in the last case signifying that the characteristic alone is negative.

Thus $\bar{3}{\cdot}7399835$ and $-3{\cdot}7399835$ are different: in fact

$$\bar{3}{\cdot}7399835 = -3 + {\cdot}7399835 = -2{\cdot}2600165.$$

58. In the tables in common use the mantissæ of the logarithms of all numbers from 1 to 100000, calculated to seven decimal places, are arranged in order. Thus when we have any number containing only five significant digits we find the mantissa of its logarithm directly from the tables, and prefix the characteristic by inspection; and if we have a logarithm, we can find the logarithm nearest to it in the tables, and so find the corresponding number correctly to five significant digits.

Numbers of more than five digits must be taken from the tables by the use of proportional parts, for an explanation of which the student is referred to Section IX. in this treatise. We will only observe that in determining the side of a triangle of about a mile in length to five digits, the error cannot exceed one inch; a degree of accuracy which is fully equal to that of the ordinary instruments for surveying.

59. Since the Trigonometrical Ratios of angles less than 90^0 are real positive quantities, they also can have logarithms, just in the same way as other quantities. The logarithms of the sines, cosines, tangents, cotangents, secants, cosecants of all angles are arranged in ordinary tables at intervals of one minute. In some tables they are given for every second. They are calculated to seven decimal places as for common numbers; but since they are mostly negative 10 is added to them, to make them positive: hence the tabular logarithm of any Trigonometrical Ratio of an angle is equal to its real logarithm increased by 10. Thus

$$L \sin \alpha = \log \sin \alpha + 10,$$

where $L \sin \alpha$ stands for the tabular logarithm, and $\log \sin \alpha$ for the real logarithm of $\sin \alpha$; and so also for the other Ratios.

60. Also since $\sin \alpha = \cos (90 - \alpha)$, therefore

$$\log \sin \alpha = \log \cos (90 - \alpha).$$

Hence if we have the log-sines of all angles from 0^0 to 45^0, we also have the log-cosines of all angles from 90^0 to 45^0: and if we have the log-cosines of all angles from 0^0 to 45^0, we have also the log-sines of all angles from 90^0 to 45^0. Hence a complete table of log-sines and log-cosines from 0^0 to 45^0 is also a complete table to 90^0. This is taken advantage of in the arrangement of the tables. The column headed sines and counted downwards from the top of the page is headed cosines at the bottom of the page and counted upwards, as may be best understood by inspecting the tables. The same arrangement applies to the tangents and cotangents, the secants and cosecants.

EXAMPLES. (A).

1. Shew that $\log 2 + \log 3 + \log 5 = \log 30$.

2. Shew that $\log 8 + \log 25 = 2 + \log 2$.

3. Shew that $\log \frac{144}{35} = 5 \log 2 + 2 \log 3 - \log 7 - 1$.

4. Shew that $\log \sin 30^\circ = -\log 2$; $\log \cos 30^\circ = \frac{1}{2}\log 3 - \log 2$; $\log \sin 45^\circ = -\frac{1}{2}\log 2$.

5. What will be the characteristics of the logarithms of the following numbers, 2750, 27·564, 28325·7, ·000042?

6. Find the value of $\sqrt[5]{30901 \cdot 7}$. *Ans.* 7·90672.

7. Find the value of $(3727593)^{\frac{2}{5}}$. *Ans.* 425·1787.

8. Find the value of $(401 \cdot 3116)^{\frac{2}{5}}$, and verify the result by arithmetic. *Ans.* 11.

9. Find the volume of a cube whose edges are 7·39148 inches. *Ans.* 403·826 cubic inches.

10. Find the volume of a rectangular parallelopiped whose edges are 4·23665, 7·39148, 3·652281 inches respectively. *Ans.* 114·3716 cubic inches.

11. How many figures will 2^{40} contain? *Ans.* 13.

12. Given log 2 find $\log \sqrt[14]{\frac{2^{60}}{5^{20}}}$. *Ans.* ·2916.

13. What is the characteristic of 476 to base 8? *Ans.* 2.

14. What is the characteristic of ·0156 to base 3? *Ans.* -4.

EXAMPLES. (B).

1. Find the logarithms of

309·017, ·0000309017, 309017000.

2. Find the logarithms of all numbers from 1 to 10.

3. Find the logarithms of the Trigonometrical Ratios of 30° and 45°.

4. Find the logarithms of 12, $\sqrt{(45)}$, $\frac{\sqrt{2}}{\sqrt[3]{3}}$.

Ans. 1·0791812, ·8266062, $\bar{1}$·9914746.

5. Find the value of $\frac{\sqrt[3]{3}}{\sqrt[4]{4} \cdot \sqrt[5]{5}}$. *Ans.* ·739148.

6. Find the value of $\frac{3}{7}\left(\frac{5}{2}\right)^{\frac{3}{2}}$. *Ans.* 1·69408.

7. Find the value of $(\cdot 0035)^{\frac{1}{7}}$. *Ans.* ·445813.

8. If $20^x = 100$, find x. *Ans.* 1·537...

9. If $7^x = 2$, find x. *Ans.* ·356...

10. Find the value of $\log \frac{1}{8}$ in terms of $\log 25$.

Ans. $\frac{3}{2}(\log 25 - 2)$.

11. Given $\log 2$, find $\log \cdot 00016$, $\log (\cdot 000016)^{\frac{1}{9}}$.

Ans. $\bar{4}\cdot 20412$, $\bar{1}\cdot 4671244$.

12. Find the number of cyphers between the decimal point and the first significant digit of $\frac{1}{5^{40}}$. *Ans.* 27.

13. Find $L \sin 36^\circ$, $L \tan 18^\circ$.

Ans. 9·7692187, 9·5117760.

EXAMPLES. (C).

1. Find the value of $\left(\frac{1}{3}\right)^{\frac{187}{14}}$. *Ans.* ·000000423665.

2. Find the value of $\left(\frac{1}{100}\right)^{\frac{19}{7}}$. *Ans.* ·000003727593.

3. Find the value of $\sin 18^\circ$; and shew that the value found above (page 47) is equal to it.

4. If $a^x = \epsilon^z$, find the fraction $\frac{x}{z}$.

5. Shew that $\log_a x : \log_b x = \log_c b : \log_c a$.

6. Transform $\sqrt{(3 \sin \alpha)} = \frac{1}{5}\sqrt{\left(\frac{\cos^3 \beta}{\sin \gamma}\right)}$,

and $$4\sqrt[4]{\left\{\left(\frac{\sin \gamma}{\cos \beta}\right)^3\right\}} = 3\sqrt[3]{\left(\frac{\sin^2 \beta}{\cos^4 \gamma}\right)},$$

into equations between the tabular logarithms of the quantities involved.

7. Assuming that log 250 and log 256 differ by ·0103, shew that $\log 2 = \cdot 30103$.

8. Find the logarithm of 9 to base $3\sqrt{3}$, and of 125 to base $\sqrt{5} \,.\, \sqrt[3]{5}$.

Ans. $1\cdot\dot{3}$, $3\cdot 6$.

9. Given $\log_{10} 2$, $\log_{10} 3$, find log 16 to base 15.

Ans. $1\cdot 023832$.

10. Given $\log_{10} 2$, $\log_{10} 3$, find $\log_5 3240$. *Ans.* $5\cdot 02244$.

11. Find $\log \dfrac{10\sqrt[3]{\cdot 045}}{\sqrt[6]{720}}$. *Ans.* $\bar{1}\cdot 9796043$.

12. Given log 2, log 3, find $\log 10\cdot\dot{6}$. *Ans.* $1\cdot 0280287$.

13. Given $\log 6 = a$, $\log 15 = b$, find log 8, log 9.

Ans. $\log 8 = \frac{3}{2}(a - b + 1)$, $\log 9 = a + b - 1$.

14. If a series of numbers are in G.P., shew that their logarithms are in A.P.

15. If x, y are the logarithms of two numbers M, N, shew that $\log \sqrt{MN} = \frac{1}{2}(x + y)$. Hence shew that $1\cdot 5$ is the logarithm of $31\cdot 622\ldots$

16. Shew (from last question) that $x + \dfrac{r(y - x)}{n}$ is the logarithm of $M\left(\dfrac{N}{M}\right)^{\frac{r}{n}}$.

17. If in last question $N = M + 1$, a number of 5 digits, and $n = 10$, $r < 10$, shew that the equation $M\left(\dfrac{N}{M}\right)^{\frac{r}{n}} = M + \dfrac{\cdot r}{10}$ is true to four decimal places.

18. If $\log 48753 = 4\cdot 6880013$, $\log 48754 = 4\cdot 6880103$, shew that $\log 487537 = 5\cdot 6880076$.

19. Prove that if the bases of different systems of logarithms increase in G.P. their moduli to any base decrease in H.P.

SECTION VII.

APPLICATION OF LOGARITHMS TO THE SOLUTION OF TRIANGLES. ILLUSTRATION.

61. We have seen that when we know the logarithms of any numbers we can at once find the logarithm of their product and quotient, but that we cannot apply logarithms directly to the determination of the sum or difference of numbers. Hence in calculating any parts of a triangle from the others by logarithms we must use the formulæ which connect the parts by means of *factors* only; if we wish to use a formula involving *terms* we must first transform it into the form of a product.

Thus in calculating a from the formulæ

$$\cos a = \frac{b^2 + c^2 - a^2}{2bc},$$

we cannot apply logarithms directly, but if we transform this equation to

$$\cos \frac{a}{2} = \sqrt{\left(\frac{s \, . \, s - a}{bc}\right)}$$

logarithms become immediately applicable.

We may observe by the way, that in using the first equation we should have to perform four operations in multiplication and one in division; in using the latter we should have to look out five logarithms. If a, b, c were simple numbers the first operation would probably give the least trouble, but as in practice they are generally quantities of five or six digits, the latter is much the shorter.

62. In most cases of solution of triangles the formulæ to be employed are in the form of products: the case which offers most difficulty is when two sides are given and the included angle; this we accordingly proceed to discuss.

Let a, b, γ be given; it is required to find the other angles α, β, and the side c.

Since $$\tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\cot\frac{\gamma}{2}, \quad \text{Art. 47},$$

we have

$$\log\tan\frac{\alpha-\beta}{2}=\log(a-b)-\log(a+b)+\log\cot\frac{\gamma}{2},$$

(supposing $a>b$) which gives $\frac{\alpha-\beta}{2}$; $\frac{\alpha+\beta}{2}$ is already known, and therefore we can easily determine α and β.

Having found α and β we can find c from the equation

$$c=a\frac{\sin\gamma}{\sin\alpha}$$

or $$\log c=\log a+\log\sin\gamma-\log\sin\alpha.$$

63. It sometimes happens that in solving this triangle we may know $\log a$, $\log b$, but not a and b, and it becomes an object to save the trouble of looking them out in the tables.

$$\tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\cot\frac{\gamma}{2}$$

$$=\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\cot\frac{\gamma}{2}.$$

Let $\frac{b}{a}=\tan\phi$, an assumption which is always allowable, since the tangent of an angle may have any value;

then $$\tan\frac{\alpha-\beta}{2}=\frac{1-\tan\phi}{1+\tan\phi}\cot\frac{\gamma}{2}$$

$$=\tan(45^\circ-\phi)\cot\frac{\gamma}{2}. \quad \text{See Art. 35 } bis\text{, 3.}$$

Now $$\log\tan\phi=\log b-\log a,$$

and is known at once, and therefore ϕ may be found from the tables; thus we know $45^\circ - \phi$, and can determine $\frac{\alpha-\beta}{2}$ from the equation

$$\log \tan \frac{\alpha-\beta}{2} = \log \tan (45^\circ - \phi) + \log \cot \frac{\gamma}{2}.$$

This method requires four logarithms to be looked out; if we had found a and b first and proceeded by the former method we should have had to look out $\log a$, $\log b$, $\log (a+b)$, $\log (a-b)$, $\log \cot \frac{\gamma}{2}$, $\log \tan \frac{\alpha-\beta}{2}$; in all six logarithms instead of four.

In these formulæ we have used the natural logarithms and not the tabular logarithms of the Trigonometrical Ratios: we must in all cases substitute $L \tan \frac{\alpha-\beta}{2} - 10$ for $\log \tan \frac{\alpha-\beta}{2}$, and similarly for the others.

63. *bis.* In the last article we found an angle ϕ from the relation $\tan \phi = \frac{b}{a}$, and by its help calculated the value of $\frac{\alpha-\beta}{2}$: now the value of ϕ itself was not required, but was only used as a means of determining $\frac{\alpha-\beta}{2}$. Angles employed in this manner are called *subsidiary angles.* As another example of their use, suppose we wish to find the cube root of $a^3 - ab^2$, where a and b are numbers containing five or six digits, and the arithmetical process would be very tedious. We proceed as follows:

$$\sqrt[3]{a^3 - ab^2} = a \sqrt[3]{1 - \frac{b^2}{a^2}}.$$

If $a > b$, let $\sin \phi = \frac{b}{a}$, which is always possible, then ϕ is determined from the equation

$$\log \sin \phi = \log b - \log a,$$

and $$\sqrt[3]{a^3 - ab^2} = a (\cos \phi)^{\frac{2}{3}},$$

or $$\log \sqrt[3]{a^3 - ab^2} = \log a + \frac{2}{3} \log \cos \phi,$$

which gives the required cube root. By this method we should have to look out five logarithms (two of which, $\log\sin\phi$, $\log\cos\phi$ would be in the same page) instead of performing four tedious operations in multiplication and one in cube root. It is left to the student to examine the modifications necessary when $b > a$.

64. We will illustrate the use of the method of Art. 63 by the example in Art. 50.

Let
$$\alpha = 37^\circ.\,30',$$
$$\alpha' = 28^\circ.\,40',$$
$$\beta = 45^\circ.\,25',$$
$$\beta' = 75^\circ.\,36',$$
$$\gamma = 16^\circ.\,28',$$
$$a = 375{\cdot}98 \text{ yards.}$$

Also let
$$\angle APQ = P,\ \angle AQP = Q.$$

Then our calculations will be as follows:

$$\begin{aligned}\log x &= \log a + L\sin\beta - L\sin(\alpha+\beta)\\ &= \log 375{\cdot}98 + L\sin 45^\circ.\,25' - L\sin 82^\circ.\,55'\\ &= 2{\cdot}5751647 + 9{\cdot}8526204 - 9{\cdot}9966727\\ &= 2{\cdot}4311124.\end{aligned}$$

Again, observing that
$$\log\sin(\alpha'+\beta') = \log\sin\{180^\circ - (\alpha'+\beta')\},$$
we have
$$\begin{aligned}\log y &= \log a + L\sin\beta' - L\sin(\alpha'+\beta')\\ &= 2{\cdot}5751647 + 9{\cdot}9861369 - 9{\cdot}9863952\\ &= 2{\cdot}5749064.\end{aligned}$$

Now
$$L\tan\frac{P-Q}{2} = L\tan(45^\circ - \phi) + L\cot\frac{\gamma}{2} - 10,$$
where
$$\begin{aligned}\log\tan\phi &= \log x - \log y\\ &= -{\cdot}1437940,\end{aligned}$$
and therefore
$$\begin{aligned}L\tan\phi &= 9{\cdot}8562060\\ &= L\tan 35.\,41'.\end{aligned}$$

Hence $$45^\circ - \phi = 9^\circ.\ 19',$$

and $$L \tan \frac{P-Q}{2} = 9\cdot2149894 + 10\cdot8395431 - 10$$
$$= 10\cdot0545325$$
$$= L \tan 48^\circ.35'.$$

Hence $$P - Q = 97^\circ.10',$$
$$P + Q = 163^\circ.\ 32';$$
$$\therefore Q = 33^\circ.\ 11'.$$

Now $$PQ = \frac{\sin\gamma}{\sin Q} x;$$

$$\therefore \log PQ = L \sin\gamma - L \sin Q + \log x$$
$$= 9\cdot4524879 - 9\cdot7382412 + 2\cdot4311124$$
$$= 2\cdot1453591$$
$$= \log 139\cdot75.$$

And therefore $PQ = 139\cdot75$ yards, the distance required.

EXAMPLES. (A).

1. Adapt the formulæ for the solution of Case I, Art. 45, to tabular logarithms.

2. Adapt the formulæ for the solutions of Cases II, III, IV, to tabular logarithms.

3. If $a = 22$, $b = 23$, $c = 25$, then $\beta = 58^\circ.\ 10'.\ 43''$.

4. If $a = 230$, $b = 240$, $c = 12$, then $\beta = 145^\circ.\ 37'.\ 35''$.

5. If $a = 23$, $\beta = 18^\circ$, $\gamma = 23^\circ.\ 42'.\ 43''$,
then $b = 10\cdot68185$, $c = 13\cdot9009$.

6. If $b = 159\cdot0643$, $\beta = 62^\circ.\ 6'.\ 51''$, $\gamma = 53^\circ.\ 27'.\ 20''$,
then $a = 162\cdot335$, $c = 144\cdot58$.

7. If $a - b = 4013\cdot116$, $a + b = 7906\cdot72$, $\gamma = 36^\circ$,
then $\alpha = 129^\circ.\ 22'.\ 26''\cdot8$, $\beta = 14^\circ.\ 37'.\ 33''\cdot2$.

8. If $a = 2820{\cdot}9385$, $b = 1430{\cdot}8485$, $\gamma = 157^\circ.\ 27'.\ 40''$,
then $\alpha = 14^\circ.\ 59'.\ 49''{\cdot}03$, $\beta = 7^\circ.\ 32'.\ 30''{\cdot}97$.

9. If $\log a = 3{\cdot}2978321$, $\log b = 4{\cdot}0135140$, $\gamma = 36^\circ.\ 52'.\ 12''$,
then $\beta = 135^\circ.\ 21'.\ 29''{\cdot}59$, $\alpha = 7^\circ.\ 46'.\ 18''{\cdot}41$.

10. Find the value of $\sqrt{(16233{\cdot}5)^2 - (10681{\cdot}85)^2}$ by means of a subsidiary angle. *Ans.* $12223{\cdot}9$.

EXAMPLES. (B).

1. If the sides of a triangle are 32, 40, 66, find the greatest angle. *Ans.* $132^\circ.\ 34'.\ 34''$.

2. The sides a, b, c of a triangle are in the ratio 4, 5, 6; find β. *Ans.* $\beta = 55^\circ.\ 46'.\ 18''$.

3. If $\alpha = 45^\circ$, $a = 14000$, $b = 15906{\cdot}43$, find the other angles.
Ans. $\beta = 53^\circ.\ 27'.\ 20''$, $\gamma = 81^\circ.\ 32'.\ 40''$,
or $\beta = 126^\circ.\ 32'.\ 40''$, $\gamma = 8^\circ.\ 27'.\ 20''$.

4. Two sides of a triangle are to each other as 9 : 7, and the included angle is $64^\circ.\ 12'$; find the other angles.
Ans. $69^\circ.\ 10'.\ 10''$, $46^\circ.\ 37'.\ 50''$.

5. If $\alpha = 23^\circ.\ 42'.\ 43''$, $\beta = 18^\circ$, $a = 207$, find b.
Ans. $b = 159{\cdot}0643$.

6. If $\alpha = 60^\circ$, $\beta = 57^\circ.\ 53'.\ 9''$, $c = 3727{\cdot}593$, find a.
Ans. $a = 3652{\cdot}28$.

7. If $\log a$, $\log b$ are given, find $\log(a + b)$ by means of a subsidiary angle.

Find also $\log(a - b)$, where $a > b$.

8. Shew how to find the value of the following expressions by means of subsidiary angles:

$$\sqrt{(a^2 \pm b^2)},\quad a \pm \sqrt{(a^2 - b^2)},\quad \sqrt{(a + b)} \pm \sqrt{(a - b)}.$$

9. Solve by means of a subsidiary angle

$$\sin x - a \operatorname{vers} x = b.$$

Ans. $\sin(x + \theta) = (a + b)\cos\theta$, where $\tan\theta = a$.

EXAMPLES. (C).

1. If two sides of a triangle are 70, 35, feet respectively, and the included angle is 36°. 52′. 12″, find the remaining angles.

Ans. 116°. 33′. 54″, 26°. 33′. 54″.

2. If the angles of a triangle be in arithmetical progression, and the greatest side is to the least in the ratio 5 : 4, find the angles. *Ans.* 70°. 53′. 36″, 49°. 6′. 24″.

3. One angle of a triangle is 60°, and the ratio of the side opposite to it to the difference of the sides containing it, is $9\sqrt{3} : 2$; find the remaining angles.

Ans. 66°. 22′. 45″, 53°. 37′. 15″.

4. If $x^2 = a^2 + b^2 + ab$, shew that subsidiary angles may be used to determine x from $x = \sqrt{(ab)} \sec\phi$, where

$$\tan\theta = \frac{b}{a},$$

and

$$\tan^2\phi = 2\,\text{cosec}\,(2\theta).$$

5. If a, b, γ be given, shew that

$$\tan\beta = \frac{2b}{a-b}\tan\frac{\gamma}{2}\cos^2\phi,\quad \tan\alpha = -\frac{2a\sec 2\phi}{a-b}\tan\frac{\gamma}{2}\cos^2\phi,$$

where

$$\tan\phi = \sqrt{\frac{a+b}{a-b}}\tan\frac{\gamma}{2}.$$

6. If

$$\sin\theta\sqrt{1+\tan^2\alpha\tan^2\beta} + \cos\theta\sqrt{1-\tan^2\alpha\tan^2\beta} = \tan\alpha + \tan\beta,$$

find θ in a form adapted to logarithmic computation.

Ans. $\sin(\theta+\phi) = \dfrac{\sin(\alpha+\beta)}{\sqrt{2}\,.\cos\alpha\cos\beta}$, where $\cos 2\phi = \tan^2\alpha\tan^2\beta$.

SECTION VIII.

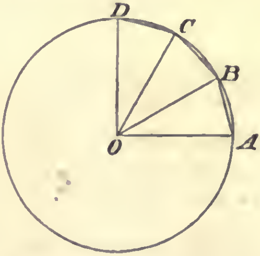
CIRCULAR MEASURE. POLYGONS INSCRIBED, AND CIRCUMSCRIBED ABOUT A CIRCLE. AREA OF A CIRCLE.

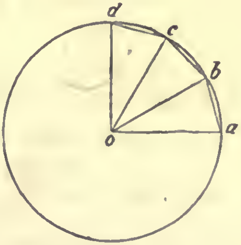
65. THERE is another system of measuring angles besides that by degrees or grades, which on account of its importance in all the higher branches of mathematics, it will be necessary to consider in this treatise.

We must premise the following propositions.

The ratio of the circumference of a circle to its diameter is the same for all circles.

To prove this, we shall assume, what is proved by Newton and appears almost self-evident, that if in a curved figure a polygon be inscribed, and the number of sides of the polygon be increased and the magnitude of each side diminished without limit, the perimeter of the polygon becomes more and more nearly equal to that of the curvilinear figure, and is ultimately equal to it.





Let *ABCD*, *abcd* be two circles whose centres are O, o : and in *ABCD* let a polygon *ABCD*... be inscribed; in *abcd* let a similar polygon *abcd*... be inscribed, then we have

$$AB : AO = ab : ao,$$
$$BC : AO = bc : ao,$$
$$CD : AO = cd : ao,$$
$$\&c... \qquad = \&c....$$

Hence $AB+BC+CD+\ldots.\ :\ AO=ab+bc+cd+\ldots.\ :\ ao$, or Perimeter of polygon $ABCD\ldots : AO =$ Per. of pol. $abcd : ao$. Now this proportion is true however many sides the polygons have; let their number be increased, and the magnitudes of each of them diminished indefinitely; then the perimeters of the polygons become the circumferences of the circles, and we have

$$\text{circumference } ABCD : AO = \text{circum. } abcd : ao,$$

or the ratio of the circumference of a circle to its diameter is the same for all circles.

This ratio is generally denoted by the symbol π; so that if r be the radius of a circle, and $2r$ its diameter, its circumference is $2\pi r$. The value of π has been calculated, and is found to be $3{\cdot}14159\ldots$ or the fraction $\frac{22}{7}$ nearly.

66. We must next prove that the angle subtended at the centre of any circle by an arc equal to the radius of the circle is of invariable magnitude.

Let AOP (fig. Art. 7) be an angle at the centre of the circle $ABA'B'$, such that the arc AP is equal to the radius AO: then, since angles at the centre of a circle are as the arcs on which they stand (Euclid, VI. 33),

$$\frac{\text{angle } AOP}{\text{4 right angles}} = \frac{\text{arc } AP}{\text{circumference } ABA'B'} = \frac{r}{2\pi r},$$

(since arc $AP =$ radius, and circumference $= 2\pi r$),

$$\text{or} \qquad \text{angle } AOP = \frac{\text{4 right angles}}{2\pi} = \frac{\text{2 right angles}}{\pi}.$$

Now π being invariable, and this expression independent of r, the angle AOP is the same whatever be the size of the circle.

Hence it appears that this angle is a proper standard by which to measure other angles. If this angle be taken for unity, any other angle which is θ times as great will be represented by θ. θ is called the circular measure of the angle.

67. To shew that θ, the circular measure of an angle, is equal to the arc subtended by that angle, divided by the radius.

Let AOP be the angle whose arc AP = radius AO, AOP_1 any other angle whose circular measure is θ, then

$$\theta = \frac{\text{angle } AOP_1}{\text{angle } AOP} = \frac{\text{arc } AP_1}{\text{arc } AP} = \frac{\text{arc}}{\text{radius}}.$$

Since the arc subtended by four right angles is $2\pi r$, the circular measure of four right angles is $\frac{2\pi r}{r} = 2\pi$. Hence the measure of a right angle is $\frac{\pi}{2}$

68. To pass from circular measure to degrees and grades and vice versâ, we have, as in Art. 5,

$$\frac{D}{90} = \frac{G}{100} = \frac{2\theta}{\pi}.$$

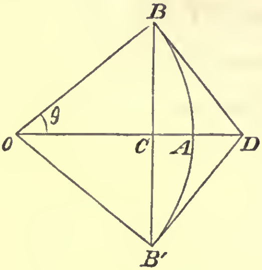
To find the number of degrees in the unit of circular measure, put $\theta = 1$ in this equation, then we have

$$D = \frac{180^\circ}{\pi} = \frac{180^\circ}{3{\cdot}14159} = 57^\circ{\cdot}29577.$$

69. It is usual to represent angles when degrees are used by the Roman letters A, B, C,..., when circular measure is used, by the Greek letters α, β, γ,...θ, ϕ,... We have used the latter throughout this treatise, even when the student had no idea of circular measure. Our reasons for so doing are mentioned in the preface: whilst we are merely dealing with the relations amongst the Trigonometrical Ratios of various angles, it is of no consequence what measure we employ; but when we trace relations between the angles themselves and their Trigonometrical Ratios, we must always use circular measure as in the following article.

70. To shew that $\frac{\sin\theta}{\theta}$, $\frac{\tan\theta}{\theta}$ are equal to unity, when $\theta = 0$.

Let $\angle AOB = \angle AOB' = \theta$, and draw BCB' perpendicular to OA, and let BAB' be the arc of a circle whose centre is O: draw BD, $B'D$ to touch this arc at B, B'.



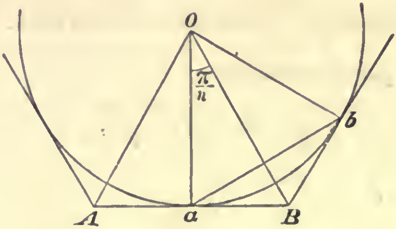
Then
$$\theta = \tfrac{1}{2}\frac{\widehat{BAB'}}{OA} = \frac{\widehat{BA}}{OA},$$
$$\tan\theta = \frac{BD}{OB} = \frac{BD}{OA},$$
$$\sin\theta = \frac{BC}{OB} = \frac{BC}{OA}.$$

Hence, since $$BD + DB' > \text{arc } BAB',$$
and $$BB' < \text{arc } BAB',$$
we have $$\tan\theta > \theta,$$
and $$\sin\theta < \theta;$$

Hence $\tan\theta$, θ, $\sin\theta$ are in descending order of magnitude; hence
$$\frac{\sin\theta}{\tan\theta}, \quad \frac{\sin\theta}{\theta}, \quad \frac{\sin\theta}{\sin\theta},$$
or $$\cos\theta, \ \frac{\sin\theta}{\theta}, \quad 1$$
are in ascending order of magnitude. But when $\theta = 0$, $\cos\theta = 1$, therefore also $\frac{\sin\theta}{\theta}$ must in this case equal unity.

Also $$\frac{\tan\theta}{\theta} = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\theta} = 1 \text{ when } \theta = 0.$$

71. To find the perimeters and areas of regular polygons circumscribed about, and inscribed in a given circle.



Let AB, ab be sides of regular polygons of n sides circumscribed about and inscribed in a circle whose centre is O and radius r.

Then $$\angle AOB = \angle aOb = \frac{2\pi}{n},$$

$$\therefore \angle aOB = \frac{\pi}{n}.$$

Then $$AB = 2aB = 2r \tan \frac{\pi}{n};$$

perimeter of circumscribed polygon $= n \,.\, AB$

$$= 2nr \tan \frac{\pi}{n};$$

area of circumscribed polygon

$$= n \,.\, aB \,.\, Oa = nr^2 \tan \frac{\pi}{n}.$$

Again, perimeter of the inscribed polygon $= 2nr \sin \frac{\pi}{n}$.

Area of inscribed polygon $= n \,.\, \Delta\, aOb = n \,.\, \frac{1}{2} aO \,.\, bO \sin aOb$

$$= \tfrac{1}{2} nr^2 \sin \frac{2\pi}{n}.$$

72. *To find the area of a circle.*

This equals the area of the inscribed polygon when the number of sides is made infinite.

Let $\frac{2\pi}{n}=\theta$, then $n=\frac{2\pi}{\theta}$, and when $n=\infty$, $\theta=0$.

The area of polygon becomes

$$\frac{\pi}{\theta}r^2\sin\theta=\pi r^2\frac{\sin\theta}{\theta},$$

when $$\theta=0,\quad \frac{\sin\theta}{\theta}=1;$$

hence, the area of the circle $=\pi r^2$.

The volume of the cylinder is the area of its base, multiplied by its height.

73. Referring the student to Art. 22, we will investigate a general expression for all the angles which satisfy the relation

$$\sin\theta=a.$$

Let AOP_1 (fig. Art. 7) or α be the smallest angle whose sine is equal to a, then all the values of θ are bounded either by OP_1, or OP_2, that is, θ is equal to α, or $(\pi-\alpha)$ taken positively, or these angles increased or diminished by any multiple of 2π; or again, θ is equal to $-(\pi+\alpha)$, $-(2\pi-\alpha)$ taken negatively, or these angles increased or diminished by any multiple of 2π. Hence all the values of θ are included in the forms $2n\pi+\alpha$, $2n\pi+(\pi-\alpha)$, $2n\pi-(\pi+\alpha)$, $2n\pi-(2\pi-\alpha)$, where n is any integer positive or negative.

Now when the coefficient of π in these forms is even (as in the first and fourth), α has the positive sign, and when odd (as in the second and third) the negative sign before it. Hence the form $m\pi+(-1)^m\alpha$, where m is any integer positive or negative, includes all these forms, since when m is even $(-1)^m$ is positive, and when m is odd $(-1)^m$ is negative; hence $\theta=m\pi+(-1)^m\alpha$.

74. To find all the values of θ which satisfy the equation

$$\cos\theta=a.$$

Let AOP_1 or a be the smallest of these values. Then all the values of θ must be bounded either by OP_1 or OP_4, that is, in the positive direction they will be a and $(2\pi - a)$, and these angles increased or diminished by any multiple of 2π; and in the negative direction they will be $-a$, and $-(2\pi - a)$, and these angles increased or diminished by any multiple of 2π. Hence all the values of θ are included in the forms $2n\pi + a$, $2n\pi + (2\pi - a)$, $2n\pi - a$, $2n\pi - (2\pi - a)$. Observing in these forms that the coefficients of π are all even, whilst a is either positive or negative, we find as the general value of θ, that $\theta = 2m\pi \pm a$, when m is any integer positive or negative.

75. To find all the values of θ which satisfy the equation

$$\tan\theta = a.$$

In the same way as in the last two Articles, we may shew that all the values in the positive direction are included in the forms $2n\pi + a$, $2n\pi + (\pi + a)$, and those in the negative direction in the forms $2n\pi - (\pi - a)$, $2n\pi - (2\pi - a)$; since in these forms a is always positive, all the values of θ are included in the form

$$\theta = m\pi + a.$$

It is not necessary in these cases that a should be the *smallest* value of θ.

76. By the help of these forms we may explain the double value of $\sin\frac{\theta}{2}$ when expressed in terms of $\cos\theta$ (see Art. 34).

Let a be one value of θ which satisfies the equation, $\cos\theta = a$,

$$\text{then } \theta = 2n\pi \pm a, \text{ and } \frac{\theta}{2} = n\pi \pm \frac{a}{2}.$$

Now if the value found for $\frac{\theta}{2}$ is as general as the value of θ or $\cos^{-1}a$ from which it is derived, $\sin\frac{\theta}{2}$ must contain all the values of $\sin\left(n\pi \pm \frac{a}{2}\right)$.

Now $\sin\left(n\pi \pm \frac{\alpha}{2}\right) = \sin n\pi \cos\frac{\alpha}{2} \pm \cos n\pi \sin\frac{\alpha}{2} = \pm \sin\frac{\alpha}{2}$, (since $\sin n\pi = 0$; $\cos n\pi = +1$ or -1, as n is even or odd).

Hence $\sin\frac{\theta}{2}$ must have two values, equal and of opposite signs.

Similarly we may shew that $\cos\frac{\theta}{2}$ when expressed in terms of $\cos\theta$ will have two values, $\pm\cos\frac{\alpha}{2}$.

77. The four values of $\sin\frac{\theta}{2}$ when expressed in terms of $\sin\theta$ (Art. 33) may be thus explained.

Let α be one value of θ which satisfies the equation, $\sin\theta = a$: then $\theta = n\pi + (-1)^n\alpha$, and $\frac{\theta}{2} = \frac{n\pi}{2} + (-1)^n\frac{\alpha}{2}$.

Now $\sin\left(\frac{n\pi}{2} + (-1)^n\frac{\alpha}{2}\right) = \sin\frac{n\pi}{2}\cos(-1)^n\frac{\alpha}{2} + \cos\frac{n\pi}{2}\sin(-1)^n\frac{\alpha}{2}$.

If n be even, $\sin\frac{n\pi}{2} = 0$, $\cos\frac{n\pi}{2} = \pm 1$, $\sin(-1)^n\frac{\alpha}{2} = \sin\frac{\alpha}{2}$, and therefore $\sin\frac{\theta}{2} = \pm\sin\frac{\alpha}{2}$.

If n be odd, $\sin\frac{n\pi}{2} = \pm 1$, $\cos(-1)^n\frac{\alpha}{2} = \cos\frac{\alpha}{2}$, $\cos\frac{n\pi}{2} = 0$, and therefore $\sin\frac{\theta}{2} = \pm\cos\frac{\alpha}{2}$.

Hence $\sin\frac{\theta}{2}$ will have four values, two and two of opposite sign.

Similarly $\cos\frac{\theta}{2} = \cos\frac{n\pi}{2}\cos(-1)^n\frac{\alpha}{2} - \sin\frac{n\pi}{2}\sin(-1)^n\frac{\alpha}{2}$

$= \pm\cos\frac{\alpha}{2}$, when n is even,

$= \pm\sin\frac{\alpha}{2}$, when n is odd.

Similar generalizations apply to all the formulæ proved in Chapters II. and III.

78. Example. $\sin\frac{\theta}{3}$ will have three values when expressed in terms of $\sin\theta$ from the cubic equation

$$4\sin^3\frac{\theta}{3} - 3\sin\frac{\theta}{3} + \sin\theta = 0, \text{ (see Art. 32).}$$

This will also appear as follows:

$\sin\frac{\theta}{3}$ must contain all the values of $\sin\left(\frac{n\pi}{3} + (-1)^n\frac{\alpha}{3}\right)$.

Now n is of the form $3m$ or $3m \pm 1$.

Let $n = 3m$, then

$$\sin\frac{\theta}{3} = \sin\left(m\pi + (-1)^{3m}\frac{\alpha}{3}\right) = \cos m\pi \sin(-1)^{3m}\frac{\alpha}{3} = \sin\frac{\alpha}{3},$$

whether m be even or odd.

Let $n = 3m \pm 1$, then

$$\sin\frac{\theta}{3} = \sin\left(m\pi \pm \frac{\pi}{3} + (-1)^{3m\pm1}\frac{\alpha}{3}\right)$$

$$= \cos m\pi \sin\left(\pm\frac{\pi}{3} + (-1)^{3m\pm1}\frac{\alpha}{3}\right) = \sin\left(\pm\frac{\pi}{3} - \frac{\alpha}{3}\right),$$

whether m be even or odd.

Hence $\sin\frac{\theta}{3}$ has 3 values,

$$\sin\frac{\alpha}{3},\ \sin\left(\frac{\pi}{3} - \frac{\alpha}{3}\right), -\sin\left(\frac{\pi}{3} + \frac{\alpha}{3}\right).$$

From the form of the equation for determining $\sin\frac{\theta}{3}$, we see that the sum of these values must equal zero, and their product $\sin\alpha$; which results may be easily verified.

EXAMPLES. (A).

In the following examples the value of π is supposed to be $\frac{22}{7}$.

1. Find the number of degrees in an angle of a regular hexagon. *Ans.* 120°.

2. Find the number of degrees in an angle of a regular pentagon. *Ans.* 108°.

3. Find the number of degrees in an angle of a polygon of n sides.

Ans. $\frac{n-2}{n}.180°$.

4. How many sides has a polygon, whose angles are 135°?

Ans. 8.

5. If the area of the circumscribed polygon equals four times the area of the inscribed polygon, find the number of sides.

Ans. 3.

6. Find the circular measure of 35°, 35^g. *Ans.* $\cdot 6\dot{1}$, $\cdot 55$.

7. Find the degrees &c. in an angle whose circular measure is 1·21. *Ans.* 69°. 18′.

8. The number of sides of one polygon is four times that of another, and the angles of the first one-half as large again as those of the second. How many sides have they? *Ans.* 20, 5.

9. Find the number of grades in the unit of circular measure.

Ans. $63^g\cdot 662$.

10. A tree is 12 feet round, what is the width of the largest plank which can be cut from it? *Ans.* 3 feet $9\frac{9}{11}$ inches.

11. A tree is 18 feet in circumference, what is the thickness of the greatest square plank which can be cut from it?

Ans. 4 feet nearly.

12. A circle of three inches diameter is cut out of a circle of 5 inches diameter; shew that the area of the remainder is equal to a circle of 4 inches diameter.

13. How many cubic feet of timber are there in a tree 16 feet in circumference and 22 feet in height? *Ans.* 448.

14. A round tower is 42 feet high: its wall is 3 feet thick, and its interior 24 feet in diameter. How many bricks 9 inches long, $4\frac{1}{2}$ inches wide, and 3 inches thick, were required to build it? *Ans.* 152064.

15. Three equal circles, radii r, are placed in contact; find the area between them. *Ans.* $\left(\sqrt{3}-\frac{\pi}{2}\right)r^2$.

Find the general values of θ in the equations

16. $\cos 7\theta+\cos\theta=0$. *Ans.* $\theta=\frac{2n+1}{8}\pi$, or $\frac{2n+1}{6}\pi$.

17. $\sin 7\theta-\sin\theta=\sin 3\theta$. *Ans.* $\theta=\frac{n\pi}{3}$, or $\frac{n\pi}{2}\pm\frac{\pi}{12}$.

18. $\sin\theta=\cos n\theta$. *Ans.* $\theta=\dfrac{2m\pi\pm\frac{\pi}{2}}{n\pm1}$.

19. $\cos\theta+\cos 3\theta=\frac{1}{2}$.

Ans. $\theta=2\left(n\pi\pm\frac{\pi}{3}\right)$, $2n\pi\pm\frac{\pi}{5}$, $2n\pi\pm\frac{3\pi}{5}$.

20. $\tan 3\theta\pm\tan\theta=0$. *Ans.* $\theta=n\pi$ or $n\pi\pm\frac{\pi}{4}$.

21. $\tan^2\theta=a^2$. *Ans.* $\theta=n\pi\pm\tan^{-1}a$.

22. $\sin^2 2\theta-\sin^2\theta=\sin^2\frac{\pi}{6}$. *Ans.* $\theta=\left(n\pm\frac{1}{5}\pm\frac{1}{10}\right)\pi$.

23. $\sin^3\theta\frac{\cos 3\theta}{3}+\cos^3\theta\frac{\sin 3\theta}{3}=\frac{m}{4}$.

Ans. $\theta=\frac{1}{4}\{n\pi+(-1)^n\sin^{-1}m\}$.

EXAMPLES. (B).

1. Find the circular measure of π grades. *Ans.* $\frac{\pi^2}{200}$.

2. At what distance from the eye will a shilling whose diameter is ·91 inches exactly hide the moon whose diameter is 30′ 12″?

Ans. $\frac{4095}{151\pi}$ feet.

3. The radius of a circle is r; find the length of an arc which subtends A^0 at the centre.

Ans. $\frac{\pi r A}{180}$.

4. The radius of the Earth being 4000 miles, what is the length of 1° of the meridian? *Ans.* $69\frac{53}{63}$ miles.

5. How far does a person at the equator travel in a second by reason of the Earth's rotation about its axis?

Ans. 512 yards nearly.

6. The diameter of the Earth's orbit about the Sun being 192,000,000 miles, how far in space does the Earth travel in one second? *Ans.* 19 miles nearly.

7. The greatest square possible is cut out of a circle whose radius is r, the area of the remainder is $(\pi - 2)r^2$.

8. The area of a square inscribed in a circle : the area of an equilateral triangle inscribed in the same circle, as $8 : 3\sqrt{3}$.

9. If a_1, a_2, a_3 be respectively the sides of a regular pentagon, hexagon, and decagon inscribed in a circle, then

$$a_1^2 = a_2^2 + a_3^2.$$

10. The area of inscribed polygon of m sides : area of circumscribed polygon of n sides $= m \cos \frac{\pi}{m} : n$, if the sides of the two polygons are equal.

11. A circular sector whose angle is θ is formed into a right cone. Shew that the vertical angle of the cone is $\sin^{-1}\dfrac{\theta}{2\pi}$.

12. A regular hexagon is inscribed in a circle (radius r), and its angular points are joined two and two so as to form another hexagon; show that the area of this latter is $\dfrac{r^2\sqrt{3}}{2}$.

13. In a sector of a circle (angle $2a$, radius r) an equilateral triangle is inscribed so that one angle is at the middle point of the arc of the sector; shew that the sides of the triangle are each equal to $\dfrac{2r}{\cot a + \sqrt{3}}$.

14. If $2(\sin 2\theta + \sin 2\phi) = 1 = 2\sin(\theta + \phi)$, find θ and ϕ.

Ans. $\theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{12} \pm \dfrac{\pi}{6}$, $\phi = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{12} \pm \dfrac{\pi}{6}$.

15. If $\sin\theta + \cos\theta < \dfrac{1+\sqrt{3}}{2}$, find the limits of θ.

Ans. θ must not lie between $2n\pi + \dfrac{\pi}{6}$ and $2n\pi + \dfrac{\pi}{3}$.

16. If $\tan(\pi\cot\theta) = \cot(\pi\tan\theta)$, then

$$\tan\theta = \frac{2n + 1 \pm \sqrt{4n^2 + 4n - 15}}{4},$$

where n is any integer positive or negative.

17. If $\tan\theta + \tan m\theta + \tan n\theta = \tan\theta\tan m\theta\tan n\theta$, then $\theta = \dfrac{r\pi}{m + n + 1}$, where r is any integer.

18. Find the least values of ϕ and θ which satisfy the equations

$$\cos(\theta + 3\phi) = \sin(2\theta + \phi),\ \sin(\phi + 3\theta) = \cos 2(\theta + \phi).$$

Ans. $\theta = \phi = \dfrac{\pi}{16}$.

19. If $\sin^2\theta\tan\theta + \cos^2\theta\cot\theta = \frac{5}{4}\operatorname{cosec} 2\theta$, then

$$\theta = (3n \pm 1)\frac{\pi}{6}.$$

20. $\tan(n\cot\theta) = \cot(n\tan\theta)$, then

$$\theta = \frac{m\pi}{2} + (-1)^m \frac{2n}{(2r+1)\pi},$$

where m, r are any integers.

EXAMPLES. (C).

1. Find the length of an arc subtending an angle of 60° in a circle whose radius is 105 feet. *Ans.* 110 feet.

2. If the length of an arc of 60° is 11 feet, then the radius of the circle is 10 feet 6 inches.

3. If a right angle were divided into 80 parts, and each of these into 80, find the number of the latter parts contained in an angle whose circular measure is ·001.

Ans. $4\frac{4}{55}$.

4. Shew that $\frac{\text{vers}\,\theta}{\theta\sin\theta} = \frac{1}{2}$, when $\theta = 0$.

5. Shew that $\frac{\sin n\theta}{\sin m\theta} = \frac{n}{m}$, when $\theta = 0$.

6. A circular flat object at distance a subtends an angle of 30′, its area is $\frac{\pi^3 a^2}{(720)^2}$ very nearly.

7. If a be the angle whose arc is r_1 and radius r_2, and a the angle whose arc is r_2 and radius r_1, then

$$\frac{a}{a'} = \left(\frac{r_1}{r_2}\right)^2.$$

8. If R, r be radii of circles circumscribed about and inscribed in a regular polygon whose side is $2a$, then

$$R^2 - r^2 = a^2.$$

9. The Earth's radius being R and the height of a tower h; find how far an object at sea must be from the foot of the tower so as just to be visible from the top.

Ans. $R\cos^{-1}\dfrac{R}{R+h}$.

If h be so small that $\dfrac{h^2}{R^2}$ may be neglected, shew that this answer becomes $\sqrt{2hR}$.

10. The height of one ship is n times that of another, and the top of one is just visible from the masthead of the other at the distance of a miles. Given R the Earth's radius, shew that the height of one is given by the equation

$$\cos^{-1}\frac{R}{R+x}+\cos^{-1}\frac{R}{R+nx}=\frac{a}{R}.$$

11. The radius of the circle of latitude 60° is 2000 miles.

12. What distance is travelled in half an hour by a person situated in latitude 60°, by reason of the Earth's rotation?

Ans. $261\frac{19}{21}$ miles.

13. If two plumb-lines suspended from points at a given distance apart on the Earth's surface are inclined at an angle of m'', and at n'' when at an elevation h, then the radius of the Earth is $\dfrac{nh}{m-n}$ very nearly.

14. A lighthouse 60 feet high is just seen from the deck of a ship 12 feet above the water, how far is the ship from the lighthouse? *Ans.* 13·79 miles.

If the height of the deck be raised 1 inch, how much further off may the lighthouse be seen? *Ans.* ·0148 miles.

15. An observer from the deck of a ship 20 feet above the sea can just see the top of a lighthouse: at the masthead (80 feet above the sea) he can just see the door of the lighthouse, which is $\dfrac{1}{4}$th of the height of the top. Find the height of the lighthouse.

Ans. 80 feet.

16. A person ascending in a balloon observes the dip of the horizon α, α' at successive heights h, h'. Shew that

$$\frac{h}{h'} = \frac{(1-\cos\alpha)\cos\alpha'}{(1-\cos\alpha')\cos\alpha}.$$

17. Find the circular measure of an angle whose complement contains as many degrees, as the supplement of an angle nine times as large contains grades. *Ans.* $\frac{11\pi}{162}$.

18. The centres of two wheels (radii r, r') are at a distance a from each other: find the length of a strap passing round them both but not crossing between them.

Ans. $\pi(r+r') + 2(r-r')\sin^{-1}\frac{r-r'}{a} + 2\sqrt{a^2-(r-r')^2}$.

19. Shew that $\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\ldots\cos\frac{x}{2^n}$ is equal to $\frac{\sin x}{x}$, when n becomes infinite.

20. If with the angular points of a regular polygon of n sides (length of side a) as centres, circles are drawn so that the points where the circles meet the sides being joined form a regular polygon of $2n$ sides, shew that the radii of the circles must be

$$\frac{a}{4}\sec^2\frac{\pi}{2n} \text{ or } \frac{a}{4}\left(1+2\cos\frac{\pi}{n}\right)\sec^2\frac{\pi}{2n}.$$

21. In last question prove that area of new polygon : area of original polygon $= \cos\frac{\pi}{n} : \cos^2\frac{\pi}{2n}$.

22. In an equilateral triangle two hexagons are inscribed, one having three of its sides coinciding with sides of the triangle, and the other three of its angles bisecting the sides of the triangle. Shew that their areas are as 4 : 3.

23. Find the three values of $\cos\frac{\theta}{3}$ when determined from the equation $\cos\theta = a$.

24. Find the three values of $\tan\frac{\theta}{3}$ when determined from the equation $\tan\theta = a$, and shew that the sum of the three values found together with three times their product is equal to zero.

SECTION IX.

ON THE USE OF PROPORTIONAL PARTS IN LOGARITHMIC AND TRIGONOMETRICAL TABLES.

79. We have shewn in the preceding Sections VI. and VII. how logarithms are used for shortening calculations involving lengthy products and powers: but we were only able to use the Tables of logarithms with exactness, when the numbers or angles were found in the Tables, and in other cases we could only extract the nearest numbers or angles. The Tables generally give the logarithms of all numbers of five digits, and of all Trigonometrical Ratios of angles for every minute of the quadrant. We proceed in this section to shew how the Tables can almost always be applied with exactness to numbers of seven digits; and to angles within the hundredth part of a second, by means of the principle of proportional parts.

80. First, as regards the logarithms of numbers. If we open the Tables at random, we find, for instance, that the logarithms of all the numbers from 26629 to 26673 increase uniformly by ·0000163, from 26674 to 26794 increase by ·0000163 or ·0000162, from 26795 to 26822 uniformly by ·0000162; and generally the logarithms of successive numbers increase gradually, nearly uniformly, with a difference which slowly diminishes as the numbers increase. Now this being true for the numbers found in the Tables, it will be equally true if we insert 100 numbers between two consecutive numbers. Thus we find by the Tables

$$\log 2662900 = 6{\cdot}4253549,$$

and

$$\log 2663000 = 6{\cdot}4253712,$$

the difference between the two logarithms being ·0000163. Now if we take the 100 numbers between 2662900 and 2663000, we may conclude that the difference between each successive number is ·000000163; and if we wish to find the logarithm of any intermediate number, as log 2662937, we must multiply ·000000163 by 37, and add the result to 6·4253549 the logarithm of 2662900, i.e.

$$\begin{aligned}\log 2662937 &= \log 2662900 + {\cdot}00000163 \times 37\\ &= 6{\cdot}4253549 + {\cdot}0000060\\ &= 6{\cdot}4253609,\end{aligned}$$

omitting the figures beyond the seventh decimal place.

81. The trouble of this multiplication is saved in the Tables. In a column by the side of the logarithms in each page is placed the difference of two consecutive logarithms, and in another column the proportional parts, or tenths of these differences, as shewn in the annexed columns. Observe that the cyphers after the decimal point are omitted for brevity: the numbers 16, 33, 49, &c., are 163 multiplied by 1, 2, 3, &c., with the last figure cut off for the division by 10; and 33 is placed opposite to 2, rather than 32, because 33 is nearer to 32·6 than 32 is, and so for 49, 82, 98, 147. When we require hundredths we cut off one more figure, and increase the last figure by unity, when the figure cut off is greater than 4. Thus to multiply 163 by 37 we take for the figures required 49 + 11 = 60, which gives the same result as above.

D.	Pro. 163	
163	1	16
	2	33
	3	49
	4	65
	5	82
	6	98
	7	114
	8	130
	9	147

Our work may be arranged as follows:—

From the Tables

$$\begin{array}{lrcr} & \log 2662900 &=& 6{\cdot}4253549\\ \text{diff. for} & 30 &=& 49\\ \quad ,, & 7 &=& 11\\ \hline \therefore & \log 2662937 &=& 6{\cdot}4253609\end{array}$$

82. The reverse process of finding a number, whose logarithm is given, more exactly than can be found directly from the Tables is similar. Thus to find the number whose logarithm is 3·4253685: we find log 2662900 = 6·4253549, difference is 136, and difference for 100 is 163, omitting the cyphers. Therefore we have to add $\frac{136}{163}$ of 100, or 84 nearly, to 2662900 to obtain

the required number; therefore 6·4253685 is the logarithm of 2662984, and 3·4253685 is the logarithm of 2662·984 the required number.

Making use of the table of proportional parts the work may be exhibited as follows :—

6·4253685 is log. of required number,
6·4253549 is log. of 2662900
136
130 is diff. of 80
6 is diff. of 4
∴ required number = 2662984

In practice the characteristic of the logarithm and the decimal point of the number may be omitted till the work is complete.

83. The principle above applied to the logarithms of numbers, applies equally well to the determination of natural sines, &c. In the latter case however the change of successive sines is not so uniform. There are 90000 numbers with 5 digits, the mantissæ of whose logarithms all lie between 0 and 1, whereas the Tables give the Trigonometrical Ratios of 5400 angles, whose sines and cosines lie between 0 and 1, so that the differences of successive sines and cosines are usually larger and less uniform than the differences of the logarithms of successive numbers; and what is said of the sines and cosines applies with still greater force to the other trigonometrical ratios, since their values extend between greater limits. Hence the differences only are given in the Tables, and the proportional parts are not calculated for them, as in the logarithms of common numbers, since to do so would render the Tables inconveniently bulky.

The mode of using the Tables will be learnt from the following examples, cyphers being omitted after decimal points for brevity.

To find sin 14°. 18′. 33″·56.

From the Tables sin 14°. 19′ = ·2472809
sin 14°. 18′ = ·2469990
diff. for 1′ = 2819

$$\therefore \quad \text{required diff.} = \frac{33.56}{60} \times 2819 = 1576 \text{ nearly,}$$

$$\therefore \quad \sin 14^\circ.\ 18'.\ 33''{\cdot}56 = {\cdot}2469990 + {\cdot}0001576$$
$$= {\cdot}2471566.$$

To find cos 14°. 18′. 33″·56,

$$\cos 14^\circ.\ 18' = {\cdot}9690157$$
$$\cos 14^\circ.\ 19' = {\cdot}9689438$$
$$\text{diff. for } 1' = \qquad 719$$

$$\therefore \quad \text{required diff.} = \frac{33.56}{60} \times 719 = 402 \text{ nearly;}$$

$$\therefore \quad \cos 14^\circ.\ 18'.\ 33''{\cdot}56 = {\cdot}9690157 - {\cdot}0000402$$
$$= {\cdot}9689755.$$

Notice that in this case the difference is subtracted, because the cosine of an angle diminishes as the angle itself increases. The same observation applies to the cosecant and cotangent.

Given $\tan\theta = {\cdot}75$, to find θ.

From the Tables $\tan 36^\circ.\ 53' = {\cdot}7503665$

$$\tan 36^\circ.\ 52' = {\cdot}7499119$$
$$\text{diff. for } 1' = \qquad 4546,$$

and $\tan\theta - \tan 36^\circ.\ 52 = {\cdot}75 - {\cdot}7499119 = 881,$

$$\therefore \quad \text{required seconds} = \frac{881}{4546} \text{ of } 60'' = 11''{\cdot}63,$$

and $\theta = 36^\circ.\ 52'.\ 11''{\cdot}63.$

Given $\operatorname{cosec}\theta = 1.\dot{6} = 1{\cdot}6666667$ nearly, to find θ.

From the Tables $\operatorname{cosec} 36^\circ.\ 52' = 1{\cdot}6667920$

$$\operatorname{cosec} 36^\circ.\ 53' = 1{\cdot}6661458$$
$$\text{diff. for } 1' = \qquad 6462$$

$$\operatorname{cosec} 36^\circ.\ 52' - \operatorname{cosec}\theta = 1{\cdot}6667920 - 1{\cdot}6666667 = 1253$$

$$\therefore \quad \text{required seconds} = \frac{1253}{6462} \text{ of } 60'' = 11''{\cdot}63;$$

$$\therefore \quad \theta = 36^\circ.\ 52'.\ 11''{\cdot}63.$$

In this case since $\tan^{-1}{\cdot}75 = \operatorname{cosec}^{-1} 1{\cdot}\dot{6}$ the results ought to agree exactly. The results obtained above do agree, but if the operation had been carried further $\tan^{-1} \cdot 75$ would have been found slightly less, and $\text{cosec}^{-1} 1\dot{\cdot}6$ slightly greater than the results obtained. The Tables being exact to 7 decimal places, the true values of the Ratios are slightly in excess or defect of the values given, and the true differences likewise. The example above shews that the method employed may be generally relied upon to give a true result as far as the hundredth part of a second.

84. The same principle also applies to the determination of the logarithms of the Trigonometrical Ratios not found exactly in the Tables, and the reverse. The following examples illustrate the method.

To find $L \sec 27^\circ.\ 13'.\ 15''\cdot34$.

$$\begin{aligned} \text{From the Tables} \quad & L \sec 27^\circ.\ 14' = 10\cdot0510248 \\ & L \sec 27^\circ.\ 13' = 10\cdot0509598 \\ & \text{diff. for} \quad 1' = \qquad 650 \end{aligned}$$

$$\therefore \quad \frac{15\cdot34}{60} \times 650 = 166\,;$$

$$\begin{aligned} \therefore \quad L \sec 27^\circ.\ 13'.\ 15''\cdot34 &= 10\cdot0509598 + \cdot0000166 \\ &= 10\cdot0509764. \end{aligned}$$

In this case the difference for 1′ is so small that each hundredth of a second does not affect the result; thus $15''\cdot33$ to $15''\cdot37$ inclusive would give the same result.

Given $L \cot \theta = 10\cdot1789937$, to find θ.

$$\begin{aligned} \text{From the Tables} \quad & L \cot 33^\circ.\ 30' = 10\cdot1792171 \\ & L \cot 33^\circ.\ 31' = 10\cdot1789426 \\ & \text{diff. for} \quad 1' = \qquad 2745. \end{aligned}$$

$$\begin{aligned} \text{Also} \quad & L \cot 33^\circ.\ 30' = 10\cdot1792171 \\ & L \cot \theta \qquad = 10\cdot1789937 \\ & \text{diff.} \qquad = \qquad 2234. \end{aligned}$$

$$\text{And} \qquad \frac{2234}{2745} \text{ of } 60'' = 48''\cdot83\,;$$

$$\therefore \quad \theta = 33^\circ.\ 30'.\ 48''\cdot83.$$

85. In the preceding Articles we have explained the method of employing proportional parts: it only remains to point out that this

method may fail from two causes, either the successive differences may be too irregular, or they may be too small. The second cause of failure applies partially to the determination of the second additional figure in numbers from 85000 to 100000; and the Trigonometrical Ratios of angles very small or very nearly 90°, and their logarithms, cannot be determined by this method from one or both causes.

For a theoretical proof of the principle of proportional parts the student is referred to more advanced treatises on the subject. A proof of the principle for the logarithms of common numbers is indicated in the Examples 16, 17, 18 C. of Section VI.

EXAMPLES. (A).

1. Given $\log 1752 = 3{\cdot}2435341$, $\log 1752{\cdot}1 = 3{\cdot}2435589$, find $\log 1752{\cdot}87$. *Ans.* $3{\cdot}2435557$.

2. Given

$$\log 48664 = 4{\cdot}6872078,$$
$$\log 48665 = 4{\cdot}6872167.$$

Calculate the table of proportional parts, and by means of it determine $\log 4866{\cdot}439$, $\log 48{\cdot}66407$, and the numbers whose logs are $\bar{1}{\cdot}6872100$, $8{\cdot}6872135$.

Ans. $3{\cdot}6872113$, $1{\cdot}6872084$, ${\cdot}4866425$, 486646500.

3. Given $\sin 16^\circ.\,24' = {\cdot}2823415$, $\sin 16^\circ.\,25' = {\cdot}2826205$, find $\sin 16^\circ.\,24'.\,17''{\cdot}3$, $\cos 73^\circ.\,35'.\,22''{\cdot}5$, $\sin^{-1}{\cdot}2824907$.

Ans. ${\cdot}2824219$, ${\cdot}2825159$, $16^\circ.\,24'.\,32''{\cdot}08$.

4. Given $\cot 32^\circ.\,36' = 1{\cdot}5636564$, diff. for $1' = 10016$, find $\cot 32^\circ.\,36'.\,27''{\cdot}59$, $\cot^{-1} 1{\cdot}5628793$.

Ans. $1{\cdot}5631958$, $32^\circ.\,36'.\,46''{\cdot}55$.

5. Given $L \tan 34^\circ.\,36' = 9{\cdot}8387571$, diff. for $1' = 2702$, find $L \tan 34^\circ.\,36'.\,27''{\cdot}3$, and angle whose $L \tan$ is $9{\cdot}8388972$.

Ans. $9{\cdot}8388800$, $34^\circ.\,36'.\,31''{\cdot}11$.

6. Given $L \operatorname{cosec} 40^\circ.\,48' = 10{\cdot}1848072$, diff. for $1' = 1463$, find $L \operatorname{cosec} 40^\circ.\,48'.\,17''{\cdot}9$, and angle whose $L \sec.$ is $10{\cdot}1847832$.

Ans. $10{\cdot}1847636$, $49^\circ.\,41'.\,50''{\cdot}16$.

7. Given $a = 18$, $b = 12$, $c = 10$, find α, β, γ.

Given $\sqrt{2} = 1{\cdot}4142136$, $\tan 54^\circ.\, 44' = 1{\cdot}4140943$,
$\tan 54^\circ.\, 45' = 1{\cdot}4149673$, $\tan 15^\circ.\, 47' = {\cdot}2826573$,
$\tan 15^\circ.\, 48' = {\cdot}2829715$.

Ans. $\alpha = 109^\circ.\, 28'.\, 16''{\cdot}39$, $\gamma = 31^\circ.\, 35'.\, 10''{\cdot}8$, $\beta = 38^\circ.\, 56'.\, 32''{\cdot}81$.

8. If $a = 5$, $b = 20$, $\gamma = 90^\circ$, find α, β.

Given $\log 2$, $L \tan 75^\circ.\, 58' = 10{\cdot}6021537$,
$L \tan 75^\circ.\, 57' = 10{\cdot}6016170$.

Ans. $\alpha = 14^\circ.\, 2'.\, 10''{\cdot}48$, $\beta = 75^\circ.\, 57'.\, 49''{\cdot}52$.

9. If $a = 18$, $b = 2$, $\gamma = 55^\circ$, find α, β.

Given $\log 2$, $L \cot 27^\circ.\, 30' = 10{\cdot}2835233$,
$L \tan 56^\circ.\, 56' = 10{\cdot}1863769$, diff. for $1' = 2763$.

Ans. $\alpha = 119^\circ.\, 26'.\, 51''{\cdot}33$, $\beta = 5^\circ.\, 33'.\, 8''{\cdot}67$.

EXAMPLES. (B).

1. If $\gamma = 90^\circ$, $a = 141$, $b = 193$, solve the triangle.

Given $\log 141 = 2{\cdot}1492191$, $\log 193 = 2{\cdot}2855573$,
$\log 2390 = 3{\cdot}3783979$, $\log 2391 = 3{\cdot}3785796$;
$L \tan 36^\circ.\, 9' = 9{\cdot}8636500$, diff. for $1' = 2652$,
$L \sin 36^\circ.\, 9' = 9{\cdot}7707793$, diff. for $1' = 1729$.

Ans. $c = 239{\cdot}018$, $\alpha = 36^\circ.\, 9'.\, 2''{\cdot}7$, $\beta = 53^\circ.\, 50'.\, 57''{\cdot}3$.

2. If $a = 254$, $\beta = 16^\circ$, $\gamma = 64^\circ$, find b.

Given $\log 2{\cdot}54 = {\cdot}4048337$, $L \sin 80^\circ = 9{\cdot}9933515$,
$L \sin 16^\circ = 9{\cdot}4403381$, $\log 7{\cdot}109 = {\cdot}8518085$,
$\log 7{\cdot}11 = {\cdot}8518696$. *Ans.* $b = 71{\cdot}0919$.

3. If $b = 246{\cdot}35$, $a = 197{\cdot}63$, $\alpha = 34^\circ.\, 27'$, find β.

Given $\log 24635 = 4{\cdot}3915526$, $\log 19763 = 4{\cdot}2958529$,
$L \sin 44^\circ.\, 50' = 9{\cdot}8482180$, $L \sin 44^\circ.\, 51' = 9{\cdot}8483450$,
$L \sin 34^\circ.\, 27' = 9{\cdot}7525761$.

Ans. $\beta = 44^\circ.\, 50'.\, 27''{\cdot}3$, or $135^\circ.\, 9'.\, 32''{\cdot}7$.

4. If $a = 254{\cdot}13$, $b = 375{\cdot}16$, $\beta = 61^\circ.\ 10'.\ 4''{\cdot}2$, find α.
Given $\log 25413 = 4{\cdot}4050559$, $\log 37516 = 4{\cdot}5742165$,
$L \sin 61^\circ.\ 10' = 9{\cdot}9425171$, diff. for $1' = 695$,
$L \sin 36^\circ.\ 24' = 9{\cdot}7733614$. *Ans.* $\alpha = 36^\circ.\ 24'$.

5. If $\alpha = 53^\circ.\ 24'$, $\beta = 66^\circ.\ 27'$, $c = 338{\cdot}65$, find a.
Given $L \sin 53^\circ.\ 24' = 9{\cdot}9046168$, $\log 33865 = 4{\cdot}5297511$,
$L \sin 60^\circ.\ 9' = 9{\cdot}9381851$, $\log 31346 = 4{\cdot}4961821$,
$\log 31347 = 4{\cdot}4961960$.
Ans. $a = 313{\cdot}4605$.

6. If $a = 275{\cdot}35$, $b = 189{\cdot}28$, $c = 301{\cdot}47$, find α.
Given $\log 38305 = 4{\cdot}5832555$, $\log 8158 = 3{\cdot}9115837$,
$\log 10770 = 4{\cdot}0322157$, $\log 19377 = 4{\cdot}2872865$,
$L \tan 31^\circ.\ 45' = 9{\cdot}7915635$, $L \tan 31^\circ.46' = 9{\cdot}7918458$.
Ans. $\alpha = 63^\circ.\ 30'.\ 57''{\cdot}8$.

7. If $a = 673{\cdot}12$, $b = 415{\cdot}89$, $\gamma = 90^\circ$, find α, β.
Given $\log 67312 = 4{\cdot}8280925$, $L \tan 58^\circ.\ 17' = 10{\cdot}2090013$,
$\log 41589 = 4{\cdot}6189785$, $L \tan 58^\circ.\ 18' = 10{\cdot}2092839$.
Ans. $\alpha = 58^\circ.\ 17'.\ 23''{\cdot}92$, $\beta = 31^\circ.\ 42'.\ 36''{\cdot}08$.

8. If $a = 456{\cdot}12$, $b = 296{\cdot}86$, $\gamma = 74^\circ.\ 20'$, find α, β.
Given $L \cot 37^\circ.\ 10' = 10{\cdot}1202593$, $\log 15926 = 4{\cdot}2021067$,
$L \tan 15^\circ.\ 35' = 9{\cdot}4454352$, $\log 75298 = 4{\cdot}8767834$,
$L \tan 15^\circ.\ 36' = 9{\cdot}4459232$.
Ans. $\alpha = 68^\circ.\ 25'.\ 18''{\cdot}12$, $\beta = 37^\circ.\ 14'.\ 41''{\cdot}88$.

9. If $a = 576{\cdot}12$, $c = 873{\cdot}14$, $\gamma = 90^\circ$: find α, β.
Given $\log 57612 = 4{\cdot}7605054$, $\log 87314 = 4{\cdot}9410839$,
$L \sin 41^\circ.\ 17' = 9{\cdot}8194012$, diff. for $1' = 1438$.
Ans. $\alpha = 41^\circ.\ 17'.\ 8''{\cdot}47$, $\beta = 48^\circ.\ 42'.\ 51''{\cdot}53$.

10. If $a = 4439$, $b = 4861$, $c = 8583$, find β.
Given $\log 8{\cdot}9415 = {\cdot}9514104$, $\log 4{\cdot}439 = {\cdot}6472851$,
$\log 4{\cdot}0805 = {\cdot}6107027$, $\log 8{\cdot}583 = {\cdot}9336391$,
$L \cos 11^\circ.\ 52' = 9{\cdot}9906180$, $L \cos 11^\circ.\ 53' = 9{\cdot}9905914$.
Ans. $\beta = 23^\circ.\ 45'.\ 46''{\cdot}46$.

11. If $a = 38$, $\beta = 48^\circ$, $\gamma = 54^\circ$, find b, c.

Given $L\sin 54^\circ = 9{\cdot}9079576$, $L\sin 78^\circ = 9{\cdot}9904044$,

$L\sin 48^\circ = 9{\cdot}8710735$, $\log 38 = 1{\cdot}5797836$,

$\log 2{\cdot}8870 = {\cdot}4604468$, diff. for $1 = 150$,

$\log 3{\cdot}1429 = {\cdot}4973306$, diff. for $1 = 138$.

Ans. $b = 28{\cdot}87039$, $c = 31{\cdot}42945$.

12. One side of a triangle is double the other, and the included angle is a right angle, find the other angles.

Given $\log 2$, $L\cot 26^\circ.\ 34' = 10{\cdot}3009994$, diff. for $1' = 3159$.

Ans. $26^\circ.\ 33'.\ 54''{\cdot}19$, $63^\circ.\ 26'.\ 5''{\cdot}81$.

13. If $a = 4$, $b = 5$, $c = 6$, find β.

Given $\log 2$, $L\cos 27^\circ.\ 53' = 9{\cdot}9464040$, $L\cos 27^\circ.\ 54' = 9{\cdot}9463371$.

Ans. $\beta = 55^\circ.\ 46'.\ 16''{\cdot}16$.

14. A rope dancer wishes to ascend a tower 100 feet high by a rope 196 feet long; at what inclination must he be able to walk up the rope?

Given $\log 2$, $\log 7$, $L\sin 30^\circ.\ 40' = 9{\cdot}7076064$, diff. for $1' = 2130$.

Ans. $30^\circ.\ 40'.\ 38''{\cdot}76$.

15. If $L\sin 16^\circ.\ 26' = 9{\cdot}4516322$, $L\sin 26^\circ.\ 27' = 9{\cdot}4520603$, find $\sin 16^\circ.\ 26'.\ 46''$.

Given $\log 2831{\cdot}1 = 3{\cdot}4519552$, $\log 2{\cdot}8312 = {\cdot}4519705$.

Ans. ${\cdot}2831133$.

EXAMPLES. (C).

1. If $b = 354$, $c = 426$, $a = 49^\circ.\ 16'$, find β, γ.

Given $\log 2$, $\log 3$, $\log 13$, $L\cot 24^\circ.\ 38' = 10{\cdot}3386231$,

$L\tan 11^\circ.\ 22' = 9{\cdot}3032609$, $L\tan 11^\circ.\ 23' = 9{\cdot}3039143$.

Ans. $\beta = 53^\circ.\ 59'.\ 4''{\cdot}9$, $\gamma = 76^\circ.\ 44'.\ 55''{\cdot}1$.

2. If $a = 60^\circ$, $b = 14$, $c = 11$, find β, γ.

Given $\log 2$, $\log 3$, $L\tan 11^\circ.44' = 9{\cdot}3174299$, diff. for $1' = 6341$.

Ans. $\beta = 71^\circ.\ 44'.\ 29''{\cdot}52$, $\gamma = 48^\circ.\ 15'.\ 30''{\cdot}48$.

3. If $a = 135$, $b = 105$, $\gamma = 60^\circ$, find α, β.

Given log 2, log 3, $L \tan 12^\circ.\ 12' = 9{\cdot}3348711$,

$L \tan 12^\circ.\ 13' = 9{\cdot}3354823$.

Ans. $\alpha = 72^\circ.\ 12'.\ 58''{\cdot}8$, $\beta = 47^\circ.\ 47'.\ 1''{\cdot}2$.

4. If $a = 5$, $b = 12$, $c = 13$, find α, β, γ.

Given log 2, $L \tan 11^\circ.\ 18'.\ 30'' = 9{\cdot}3009670$,

$L \tan 11^\circ.\ 18'.\ 40'' = 9{\cdot}3010764$.

Ans. $\alpha = 22^\circ.\ 37'.\ 11''{\cdot}52$, $\beta = 67^\circ.\ 22'.\ 48''{\cdot}48$, $\gamma = 90^\circ$.

5. Two sides of a triangle are as 5 : 9, and the included angle is a right angle, find the other angles.

Given log 2, log 3, $L \tan 29^\circ.\ 3' = 9{\cdot}7446453$,

$L \tan 29^\circ.\ 4' = 9{\cdot}7449428$.

Ans. $29^\circ.\ 3'.\ 16''{\cdot}55$, $60^\circ.\ 56'.\ 43''{\cdot}45$.

6. Prove that $\tan\left(\beta + \frac{\alpha}{2}\right) = \frac{c+b}{c-b} \tan \frac{\alpha}{2}$; and if $b : c = 21 : 29$, and $\alpha = 46^\circ.\ 37'.\ 24''$, find β.

Given log 2, $L \tan 23^\circ.18' = 9{\cdot}6341426$, diff. for $1' = 3477$,

$L \tan 69^\circ.\ 37' = 10{\cdot}4299645$, $L \tan 69^\circ.\ 38' = 10{\cdot}4303516$.

Ans. $\beta = 46^\circ.\ 19'.\ 4''{\cdot}73$.

7. If $a = 140{\cdot}5$, $b = 170{\cdot}6$, $\alpha = 40^\circ$, find β, γ.

Given $L \sin 40^\circ = 9{\cdot}8080675$, $\log 1405 = 3{\cdot}1476763$,

$\log 1706 = 3{\cdot}2319790$,

$L \sin 51^\circ.18' = 9{\cdot}8923342$, diff. for $1' = 1012$.

Ans. $\beta = 51^\circ.\ 18'.\ 21''{\cdot}34$, $\gamma = 88^\circ.\ 41'.\ 38''{\cdot}66$.

8. If $a = 1{\cdot}56234$, $b = {\cdot}43766$, $\gamma = 58^\circ.\ 42'.\ 6''{\cdot}1$, find α, β.

Given $\log 56234 = 4\frac{3}{4}$, $\log \cot 29^\circ.\ 21' = {\cdot}2500150$,

$\log \cot 29^\circ.\ 22' = {\cdot}2497194$.

Ans. $\alpha = 105^\circ.\ 38'.\ 56''{\cdot}95$, $\beta = 15^\circ.\ 38'.\ 56''{\cdot}95$.

9. If $a = 3$, $b = 1$, $\gamma = 53^\circ.\ 7'.\ 48''$, find c without determining α and β.

Given log 2, $\log 25298 = 4{\cdot}4030862$, diff. for $1 = 172$,

$L \cos 26^\circ.\ 33'.\ 54'' = 9{\cdot}9515452$, $L \tan 26^\circ.\ 33'.\ 54'' = 9{\cdot}6989700$.

Ans. $c = 2{\cdot}529823$.

10. If $\tan\theta = \dfrac{2\sqrt{ab}\sin\dfrac{\gamma}{2}}{a-b}$, and $a=5$, $b=2$, $\gamma=120^\circ$, find θ.

Given log 3, $L\tan 61^\circ.\,17' = 10{\cdot}2613287$,

$L\tan 61^\circ.\,18' = 10{\cdot}2616286$.

Ans. $\theta = 61^\circ.\,17'.\,22''{\cdot}13$.

11. One angle of a triangle is 60°, and the side opposite is to the difference of the two sides including it as 9 : 2; find the remaining angles.

Given log 3, $L\cos 78^\circ.\,54'.\,10'' = 9{\cdot}2843730$,

$L\cos 78^\circ.\,54'.\,20'' = 9{\cdot}2842656$.

Ans. $71^\circ.\,5'.\,44''{\cdot}88$, $48^\circ.\,54'.\,15''{\cdot}12$.

MISCELLANEOUS EXAMPLES.

1. Prove that the ratios of the angles denoted by a French and English degree, minute, second, are expressed by

$$\frac{3\,.\,3}{2\,.\,5},\ \frac{3\,.\,3^2}{2\,.\,5^2},\ \frac{3\,.\,3^3}{2\,.\,5^3},\ \text{respectively.}$$

2. Shew that $m' = m^{\backprime} + \frac{23}{27} m^{\backprime}$, and $n'' = 3n^{\backprime\backprime} + \frac{7}{81} n^{\backprime\backprime}$.

3. Shew that the area of a triangle is equal to

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\left(\frac{a^2}{\sin\alpha}+\frac{b^2}{\sin\beta}+\frac{c^2}{\sin\gamma}\right).$$

4. One regular figure has twice as many sides as another, and an angle of the first is one-third as large again as an angle of the second; find the interior angles of each. *Ans.* 144^0, 108^0.

5. Solve the equations:

$$\sin 3\phi \sin 4\phi - \sin 2\theta \sin\phi = \frac{1}{2}\cos 4\phi,$$

$$4\sin(\phi-\theta)\cos(\phi+\theta) = \cos 6\phi \operatorname{cosec}\phi.$$

Ans. $\phi = 60^0,\ \theta = \frac{1}{2}\sin^{-1}\frac{1}{2\sqrt{3}}$.

6. If regular pentagons be inscribed and circumscribed about a circle, then perimeter of the former : perimeter of the latter $=\sqrt{5}+1 : 4$.

7. If $\cos\theta = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos A$,

and
$$\sin^2\phi = \sin\alpha\sin\beta\cos^2\frac{A}{2},$$

then
$$\sin^2\frac{\theta}{2} = \sin\left(\frac{\alpha+\beta}{2}+\phi\right)\sin\left(\frac{\alpha+\beta}{2}-\phi\right).$$

8. If $\tan x = \cos\alpha \tan y$, then

$$\tan(y-x) = \frac{\tan^2\frac{\alpha}{2}\sin 2y}{1+\tan^2\frac{\alpha}{2}\cos 2y}.$$

9. Eliminate θ between the equations

$$m = \operatorname{cosec}\theta - \sin\theta, \quad n = \sec\theta - \cos\theta.$$

Ans. $(mn)^{\frac{2}{3}}(m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1.$

10. If $\tan\frac{\alpha}{2} = \tan^3\frac{\beta}{2}$, and $\tan\beta = 2\tan\phi$,

then
$$\phi = \frac{\alpha+\beta}{2}.$$

11. If $a\cos\theta + b\cos(\theta+\alpha)$ be put in the form

$$A\cos(\theta+B),$$

find the values of A and B.

Ans. $A = \sqrt{(a^2+b^2+2ab\cos\alpha)}, \quad B = \tan^{-1}\dfrac{b\sin\alpha}{a+b\cos\alpha}.$

12. Divide a given angle α into two parts, whose sines are in the ratio $m : n$. *Ans.* $\sin^{-1}\dfrac{m\sin\alpha}{\sqrt{m^2+n^2+2mn\cos\alpha}}, \ \sin^{-1}\dfrac{n\sin\alpha}{\sqrt{m^2+n^2+2mn\cos\alpha}}.$

13. If $\cos(\phi-\alpha)$, $\cos\phi$, $\cos(\phi+\alpha)$ are in H.P., then

$$\cos\phi = \sqrt{2}\cos\frac{\alpha}{2}.$$

14. The elevation of a tower standing on a horizontal plain is observed; a feet nearer it is found to be 45°; b feet nearer still it is the complement of what it was at the first station; shew that the height of the tower is $\dfrac{ab}{a-b}$ feet.

15. The distance between the centres of the inscribed circle and of the escribed circle touching the side BC is

$$a\sec\frac{\alpha}{2}.$$

16. Prove the following expression for the area of a triangle,

$$\frac{1}{2}\left(\frac{a^2\cos\beta\cos\gamma}{\sin\alpha} + \frac{b^2\cos\alpha\cos\gamma}{\sin\beta} + \frac{c^2\cos\alpha\cos\beta}{\sin\gamma}\right).$$

17. A number of spheres are put in a hollow cone one above another, and each touching the one above, the one below, and the sides of the cone; shew that the radii of the spheres form a geometrical progression of which the common ratio is

$$\tan^2\left(45^\circ + \frac{\alpha}{4}\right),$$

α being the vertical angle of the cone.

18. From one of the angles of a rectangle a perpendicular is drawn to its diagonal, and from the point of intersection lines are drawn perpendicular to the sides which contain the opposite angle; shew that if p, p' be the perpendiculars last drawn, and d the diagonal of the rectangle,

$$p^{\frac{2}{3}}+p'^{\frac{2}{3}}=d^{\frac{2}{3}}.$$

19. If two circles whose radii are a, b touch each other externally, and if θ be the angle between their common tangents, shew that

$$\sin\theta=\frac{4\,(a-b)\sqrt{(ab)}}{(a+b)^2}.$$

20. If $(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi)$,

$$a\cos 2\phi+b\cos 2\theta=c,$$

then
$$b^2-a^2-c^2+2ac\cos 2\phi=0.$$

21. If $\cot\beta-\cot\theta=\cot(\alpha+\theta)+\cot(\alpha-\beta)$,

then
$$\frac{\sin(\alpha-\beta)}{\sin\beta}=\frac{\sin(\alpha+\theta)}{\sin\theta}.$$

22. If $\tan\frac{\alpha}{2}=2-\sqrt{3}$, find $\sin\alpha$.

23. The elevation of a tower at a place A due south of it is 30°; and at a place B, due west of A, and at the distance a from it, the elevation is 18°: shew that the height of the tower is

$$\frac{a}{\sqrt{(2\sqrt{5}+2)}}.$$

24. If $\sin\alpha+\sin(\theta-\alpha)+\sin(2\theta+\alpha)=\sin(\theta+\alpha)+\sin(2\theta-\alpha)$, then

$$\theta=\cos^{-1}\frac{\pm\sqrt{5}+1}{4}.$$

25. If $\dfrac{m\tan(\alpha-x)}{\cos^2 x}=\dfrac{n\tan x}{\cos^2(\alpha-x)}$,

then
$$\tan(\alpha-2x)=\frac{n-m}{n+m}\tan\alpha.$$

26. If a, a' be homologous sides of similar triangles inscribed in and described about a given circle, then

$$a=4a'\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

27. In the ambiguous case in the solution of triangles, if $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the angles of the triangle in the two solutions,

$$\frac{\sin\gamma}{\sin\beta}+\frac{\sin\gamma'}{\sin\beta'}=2\cos\alpha.$$

28. The elevation of two clouds to a person in the same line with them is α. When standing on the shadow of one of them its elevation is 2α, and the other is vertically over him. Shew that the heights of the clouds are as $2\cos^2\alpha : 1$.

29. At each of three stations in the same horizontal plane, and at given distances from each other, the elevation of a tower is observed to be α; find the height of the tower.

30. In any triangle

$$a\cos 2\beta + 2b\cos\alpha\cos\beta + a\cos 2\gamma + 2c\cos\alpha\cos\gamma = 0.$$

31. If $\sqrt{(a^2-x^2)}\cos\beta + a\sin\alpha = x\sin\beta$, then

$$x = \mp a\cos(\alpha \pm \beta).$$

32. The elevation of a tower on a horizontal plane is observed: on advancing a feet nearer its elevation is found to be the complement of the former; on again advancing its elevation is found to be double of its first elevation: shew that the last station is $\frac{a}{2}$ feet from the foot of the tower.

33. If the line bisecting the right angle γ of a triangle divide the hypothenuse into parts x, y, then

$$x : y = 1 - t : 1 + t, \text{ where } t = \tan\frac{\alpha-\beta}{2}.$$

34. If

$$\sin\theta + \sin 2\theta = m,$$
$$\cos\theta + \cos 2\theta = n,$$

then $(m^2+n^2)^2 - 3(m^2+n^2) - 2n = 0$.

35. $(\sin\alpha + \sec\alpha)^2 + (\cos\alpha + \operatorname{cosec}\alpha)^2 = (1 + \sec\alpha\operatorname{cosec}\alpha)^2$.

36. $\tan^{-1}\{\frac{1}{2}(\cot^2\theta - \tan^2\theta)\} = 2\tan^{-1}(\cos 2\theta)$.

37. A person observes that a tower and spire on a hill subtend the same angle β; on ascending the hill a feet, he finds the tower subtends an angle γ and the spire again subtends an angle β; shew that the height of the spire is

$$\frac{a\sin\beta}{\sin(\gamma-\beta)}.$$

38. If the diameter $2R$ of a semicircle be divided into any two parts, and on these parts semicircles be described whose radii are r_1, r_2; then the radius of the circle which touches all three semicircles is

$$\frac{Rr_1r_2}{R^2 - r_1r_2}.$$

39. Shew that $\dfrac{\sin 2\alpha+\sin 4\alpha+\sin 6\alpha}{4\cos^2\alpha\cos^2 2\alpha}=\tan\alpha+\tan 2\alpha$.

40. A person on the top of a mountain observes the depression 45° of an object on the plane below him: he then turns through an angle of 30°, and observes the depression of another object on the same plane to be 30°. On descending the mountain he finds the distance between the objects is d. Shew that the height of the mountain is also d.

41. If α, β, γ be in arithmetical progression,

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}-\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}=2\cos 2\beta.$$

42. A tower stands in a field in the form of an equilateral triangle, and subtends angles $\tan^{-1}\dfrac{4}{\sqrt{3}}$, $\tan^{-1}\dfrac{4}{\sqrt{7}}$, $\tan^{-1}\dfrac{4}{\sqrt{7}}$ at the three corners: its height is equal to a side of the field.

43. If $$\tan^{-1}\frac{1}{x-1}-\tan^{-1}\frac{1}{x+1}=\frac{\pi}{12},$$
then $$x=\sqrt{3}+1.$$

44. If $$\tan(\theta+\phi)=m\tan(\theta-\phi),$$
and $$a\cos^2\phi=b\cos^2\theta,$$
then $$\cos^2\theta=\frac{(m+1)^2b-(m-1)^2a}{(m+1)^2b^2-(m-1)^2a^2}a.$$

45. If x, y, z be the lines joining the feet of the perpendiculars from the angles of a triangle upon the opposite sides,

then $$\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=\frac{a^2+b^2+c^2}{2abc}.$$

46. If $\cos(\alpha-\beta)\sin(\gamma-\delta)=\cos(\alpha+\beta)\sin(\gamma+\delta)$,
then $$\cot\delta=\cot\alpha\cot\beta\cot\gamma.$$

47. Shew that

$$\sin\alpha\cos(\beta-\gamma)-\sin\beta\cos(\gamma-\alpha)=\sin(\alpha-\beta)\cos\gamma.$$

48. Shew that

$$\frac{\cos\alpha+\sin\gamma-\sin\beta}{\cos\beta+\sin\gamma-\sin\alpha}=\frac{1+\tan\frac{\alpha}{2}}{1+\tan\frac{\beta}{2}},\text{ if }\alpha+\beta+\gamma=90^\circ.$$

49. Shew that

$$\frac{2\operatorname{cosec}2\theta-\sec\theta}{2\operatorname{cosec}2\theta+\sec\theta}=\cot^2\left(45^\circ+\frac{\theta}{2}\right).$$

50. If $\alpha+\beta+\gamma=0$,

$$\cos\alpha+\cos\beta+\cos\gamma+1=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

51. If the area of a triangle be constant, the sum of the cotangents of the angles varies as the sum of the squares of the sides.

52. A column on a pedestal 20 feet high subtends an angle of 45° to a person on the ground; on approaching 20 feet it again subtends an angle of 45°. The height of the column is 100 feet.

53. Shew that

$$\cos^2\sigma+\cos^2(\sigma-\theta)+\cos^2(\sigma-\phi)+\cos^2(\sigma-\psi)=2+2\cos\theta\cos\phi\cos\psi,$$

where

$$\sigma=\frac{\theta+\phi+\psi}{2}.$$

54. If $\tan\theta=\cos 2\alpha$,

then

$$\sin 2\theta=\frac{1-\tan^4\alpha}{1+\tan^4\alpha}.$$

55. Shew that

$$2\tan\left\{\cos^{-1}\frac{1}{\sqrt{(1+x^2)}}-\cos^{-1}\frac{x}{\sqrt{(1+x^2)}}\right\}=x-\frac{1}{x}.$$

56. Shew that in a triangle

$$(r_a-r)(r_b-r)(r_c-r)=4Rr^2.$$

57. If $a'b'c'$ be the perpendiculars from the centre of the circumscribed circle upon the sides a, b, c respectively, then

$$\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}=\frac{1}{4}\cdot\frac{abc}{a'b'c'}.$$

58. Shew that $\cos^{-1}\frac{1}{2}-\sin^{-1}\frac{1}{5}=\tan^{-1}\left(\frac{9\sqrt{3}-8\sqrt{2}}{5}\right)$.

59. If $\alpha=\tan^{-1}\frac{1}{7}$, $\beta=\tan^{-1}\frac{1}{3}$, then

$$\cos 2\alpha=\sin 4\beta.$$

60. If $2\tan x+\tan(\alpha-x)=\tan(\beta+x)$, then

$$\tan x=\frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta}.$$

61. From the top of a tower the depressions α, β of two objects in the same horizontal plane with the foot of the tower are observed, and also the angle ω which they subtend; and the distance a between them is known; the height of the tower is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{(\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \cos \omega)}}.$$

62. Shew that

$$\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha + \gamma}{2} \sin \frac{\beta + \gamma}{2}.$$

63. If $\sin \theta + \sin \phi = m,$

and $\cos \theta + \cos \phi = n,$

then $\cos (\theta + \phi) = \dfrac{n^2 - m^2}{n^2 + m^2}.$

64. Shew that $\tan^{-1} \dfrac{x \cos \phi}{1 - x \sin \phi} - \tan^{-1} \dfrac{x - \sin \phi}{\cos \phi} = \phi.$

65. If the tangents of the half angles of a triangle be in arithmetical progression, shew that the cosines of the whole angles are so also.

66. Shew that

$$\cos^{-1} \frac{1 - m^2}{1 + m^2} - \cos^{-1} \frac{1 - n^2}{1 + n^2} = \tan^{-1} \frac{2 (m - n) (1 + mn)}{(1 + mn)^2 - (m - n)^2}.$$

67. If the straight line which bisects the angle BAC of a triangle divides the side BC in the ratio $m : n$, then

$$\tan \beta = \frac{n \sin \alpha}{m - n \cos \alpha}.$$

68. Prove that the product of the perpendiculars from the angles on the opposite sides equals

$$\frac{\{(a + b + c) r\}^3}{abc}.$$

69. If lines be drawn from the angles of a triangle to the centre of the inscribed circle, and the points where these lines meet the circle are joined so as to form a new triangle, shew that its sides are in the ratio

$$\sin \frac{\alpha}{4} + \cos \frac{\alpha}{4} : \sin \frac{\beta}{4} + \cos \frac{\beta}{4} : \sin \frac{\gamma}{4} + \cos \frac{\gamma}{4}.$$

70. Eliminate θ and ϕ from the equations

$$a \sin^2 \theta + b \cos^2 \theta = \alpha,$$
$$b \sin^2 \phi + a \cos^2 \phi = \beta,$$
$$a \tan \theta = b \tan \phi.$$

Ans. $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{\alpha} + \dfrac{1}{\beta}.$

71. Shew that the value of $a\cos 2\theta + b\sin 2\theta$, when $\tan\theta = \frac{b}{a}$, is a.

72. $\sin\alpha\sin\beta\sin(\alpha-\beta)+\sin\beta\sin\gamma\sin(\beta-\gamma)+\sin\gamma\sin\alpha\sin(\gamma-\alpha)$

$$=\frac{1}{4}\{\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)\}.$$

73. A circle is inscribed in an equilateral triangle (side a), and another circle touches this circle and two of the sides, shew that its radius is $\frac{a}{6\sqrt{3}}$.

74. AB is a tower at the foot of a hill whose inclination is θ; C, D are two stations directly up the hill from B such that $BC=CD$, and $\angle ACD=\alpha$, $\angle ADC=\beta$. Shew that θ is found from the equation

$$\cot\theta=\frac{\sin\alpha\sin\beta}{2\cos\alpha\sin\beta+\cos\beta\sin\alpha}.$$

75. If $\sin\alpha$ be the arithmetic, and $\sin\beta$ the geometric mean between $\sin\gamma$ and $\cos\gamma$, prove that

$$\cos 2\alpha=\frac{1}{2}\cos 2\beta=\cos^2(45^\circ+\gamma).$$

76. If α, β, γ are each less than 90°, then if $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=1$, any two of them are together less than 90°, and if $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, any two of them are together greater than 90°.

77. If ABC be a triangle, prove from the definitions that

$$\sin C=\sin A\cos B+\cos A\sin B.$$

78. A balloon considered as a vertical object of given height floats at a constant height above the earth, and subtends angles α, β at a place when the elevations of its lowest points are A, B respectively; shew that

$$\tan(A+\alpha)\cot A=\tan(B+\beta)\cot B.$$

79. Eliminate α and β from the equations

$$\sin\alpha\cos\beta\sin\theta+\cos\alpha\cos\theta=A,$$
$$\sin\alpha\cos\beta\cos\theta-\cos\alpha\sin\theta=B,$$
$$\sin\alpha\sin\beta\cot\theta=c.$$

Ans. $A^2+B^2=1-c^2\tan^2\theta.$

80. Find the relation between a, b, c that an angle of a degrees may be expressed by b, when the unit of measurement is $\frac{\omega}{c}$, ω being the unit of circular measurement. *Ans.* $\pi ac=180b.$

81. Eliminate θ and ϕ from the equations

$$\cos\theta\sin\alpha+\cos\phi\cos\alpha=c,$$
$$\frac{\cos(\theta+\phi)}{\cos\alpha}=\frac{\cos(\theta-\phi)}{\sin\alpha},$$
$$\frac{\cos\vartheta}{\sin\alpha}=\frac{\cos\phi}{\cos\alpha}.$$

Ans. $2(1+\sin 2\alpha)(c^2-1)=c^4\sin^3 2\alpha$.

82. If A, B represent a certain angle in two systems whose units are the angles subtended at the centre of a circle by $\frac{1}{m}$th, $\frac{1}{n}$th parts of the circumference respectively, find the relation between A, B, and express the angle in degrees. *Ans.* $An=Bm$, and angle $=\frac{A}{m}\,.\,360^0$.

83. If the sides of an acute-angled triangle of given perimeter be in A.P., the common difference of the sides cannot exceed, and the radius of the inscribed circle cannot be less than $\frac{1}{12}$th of the perimeter.

84. A, B are two objects in the same horizontal plane, P a point at which the angle α subtended by AB is observed: from P two persons walk in directions at right angles to PA, PB respectively to points Q, R, at each of which the angle subtended by AB is α: if $PQ=a$, $PR=b$, find the distance AB, α being acute. *Ans.* $\sqrt{a^2+b^2+2ab\cos\alpha}$.

85. Eliminate a, b, c from the equations

$$a=b\cos C+c\cos B,$$
$$b=a\cos C+c\cos A,$$
$$c=a\cos B+b\cos A.$$

Ans. $\cos A+\cos(B\pm C)=0$.

86. Eliminate θ from the equations

$$\operatorname{cosec}^2\theta=m\tan\theta,$$
$$\sec^2\theta=n\cot\theta.$$

Ans. $\sqrt{m}+\sqrt{n}=\sqrt[4]{m^3n^3}$.

87. The numerical measures of the angles of a quadrilateral when referred to units 1^0, 2^0, 3^0, 4^0 respectively are in A.P., and the difference between the second and fourth is 90^0; find the angles.

Ans. 30^0, 66^0. 108^0, 156^0.

88. A person walking up a slope towards the summit of a hill observes the elevations of the summit at A, B, C to be α, 2α, 3α respectively; and also finds that $AB:BC=1:n$. Shew that the inclination of the slope is

$$\tan^{-1}\frac{n\sin 3\alpha-\sin\alpha}{n\cos 3\alpha-\cos\alpha}.$$

89. If $ABCD$ be a quadrilateral figure capable of having a circle inscribed in it, and also described about it, prove that the radius of the inscribed circle is equal to

$$\frac{AB+BC+CD+DA}{2\,(\operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C+\operatorname{cosec} D)}.$$

90. If $A'B'C'$ be the centres of the escribed circles of a triangle ABC, and O the centre of the inscribed circle, shew that the radii of the circles circumscribing $A'B'C'$, $A'OB'$, $A'OC'$, $B'OC'$, are each equal to twice the radius of the circle circumscribing the triangle ABC.

91. If in last question $a'b'c'$ be the sides of the triangle $A'B'C'$, shew that

$$\frac{a'^2}{r_b+r_c}=\frac{b'^2}{r_a+r_c}=\frac{c'^2}{r_a+r_b}=\frac{(r_a+r_b)\,(r_a+r_c)\,(r_b+r_c)}{r_ar_b+r_ar_c+r_br_c}=\frac{a'b'c'}{2\sqrt{\sigma(\sigma-a')\,(\sigma-b')\,(\sigma-c')}},$$

where $2\sigma=a'+b'+c'$.

92. If $\dfrac{\sqrt{\sin(\theta+2\lambda)}}{\cos(2\theta+\lambda)}=\dfrac{\sqrt{(\beta^2-\alpha^2)\sin\theta+2\alpha\beta\cos\theta}}{\cos\theta\,(\beta\cos\theta-\alpha\sin\theta)-\sin\theta\,(\beta\sin\theta+\alpha\cos\theta)}$,

then $\tan\lambda=\dfrac{\alpha}{\beta}$.

93. In a triangle

$$\sin\alpha\sin(\alpha-\beta)\sin(\alpha-\gamma)+\sin\beta\sin(\beta-\alpha)\sin(\beta-\gamma)$$
$$+\sin\gamma\sin(\gamma-\alpha)\sin(\gamma-\beta)=\sin\alpha\sin\beta\sin\gamma-\sin 2\alpha\sin 2\beta\sin 2\gamma.$$

94. Find θ and ϕ from the equations

$$p\sin^4\theta-q\sin^4\phi=p,$$
$$p\cos^4\theta-q\cos^4\phi=q.$$

Ans. $\cos\theta=\sqrt[4]{\dfrac{q^2}{p(q-p)}}$, $\cos\phi=\sqrt[4]{\dfrac{p}{q-p}}$.

95. If α, β, γ be unequal and each less than $\pm 2\pi$, and if

$$\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0,$$

then
$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{3}{2}.$$

96. If $\dfrac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\beta-\gamma}{2}}$, $\dfrac{\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha-\gamma}{2}}$, $\dfrac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$ are in A. P., so also are $\cos 2\alpha$, $\cos 2\beta$, $\cos 2\gamma$, α, β, γ being the angles of a triangle.

97. If in a triangle ABC lines be drawn through the angular points inclined at an angle α to the three sides respectively, and $A'B'C'$ be the triangle formed by their intersections, shew that area of $A'B'C'$: area of $ABC=\sin^2(\phi-\alpha):\sin^2\phi$, where $\cot\phi=\cot A+\cot B+\cot C$.

98. If $$(1+\sin\theta)(1+\sin\phi)(1+\sin\psi)=\cos\theta\cos\phi\cos\psi,$$
shew that $$\sin\theta+\sin\phi+\sin\psi+\sin\theta\sin\phi\sin\psi=0,$$
and $$\sec^2\theta+\sec^2\phi+\sec^2\psi-2\sec\theta\sec\phi\sec\psi=1.$$

99. The four common tangents drawn to two circles (radii r, r') form a quadrilateral. If 2α, $2\alpha'$ be the inclinations of the pairs of tangents which are similarly situated, shew that the area of the quadrilateral is
$$rr'(\cot\alpha-\cot\alpha').$$

100. Shew from a figure that
$$\tan 80^0=\sqrt{3}+4\cos 10^0.$$

101. If a circle be drawn touching the sides AB, AC of a triangle ABC produced, and the circumscribed circle, shew that its radius is equal to
$$4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\sec^2\frac{\alpha}{2}.$$

102. Eliminate θ from the equations
$$\frac{x}{a\sin\theta}+\frac{y}{b\cos\theta}=1,\quad \frac{x}{a\sin^3\theta}+\frac{y}{b\cos^3\theta}=0.$$
Ans. $$\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}}-\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}}=1$$

103. If in a triangle ABC, PD be drawn from a point P in AC inclined to AC at an angle θ, and meeting AB in D, so as to bisect the triangle, then
$$\cot\theta=\frac{2AP^2}{bc}\operatorname{cosec}A-\cot A.$$

104. If $ABCD$ be a quadrilateral inscribed in a circle, and O, P the middle points of AB, CD, and if
$$POA=\theta,\ OPC=\phi,\ AB=2a,\ CD=2b,\ PO=c,$$
then $$\frac{\tan\theta}{\tan\phi}=\frac{c^2-a^2+b^2}{c^2+a^2-b^2}.$$

105. If $$\frac{\tan(x+y)}{a+b}=\frac{\tan(y+z)}{b+c}=\frac{\tan(x+z)}{a+c},$$
and $$\frac{\tan(x+y)}{a'+b'}=\frac{\tan(y-z)}{b'+c'}=\frac{\tan(x-z)}{a'+c'},$$
then $$\frac{b^2-a^2}{b^2-c^2}=\frac{b'^2-a'^2}{b'^2-c'^2}.$$

106. Eliminate θ from the equations
$$\frac{x}{a}=\cos\theta+\cos 2\theta,\quad \frac{y}{b}=\sin\theta+\sin 2\theta.$$
Ans. $$\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=\left(1+\frac{x}{a}\right)^2+\frac{y^2}{b^2}.$$

107. A, B, C is a triangle; A', B', C' the middle points of the arcs BC, AC, AB of the circumscribing circle. If chord $B'C'$ intersects AB, AC in L, M, shew that

$$AL = AM = \frac{a \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sin \alpha \cos \frac{\alpha}{2}}.$$

108. OP is a vertical pole in a circular inclosure $ADBC$, AB a diameter of the inclosure, and CD a chord through O. If 3α, α be the elevations of P at A, B and 2θ, θ at D, C respectively, shew that

$$\cos 2\theta = \cos \alpha \cos 3\alpha \sec 2\alpha.$$

109. Three lines parallel to the sides of a triangle pass through a point and cut off portions of the triangle whose areas are to the area of the triangle as $m : 1$, $n : 1$, $p : 1$ respectively: shew that

$$\sqrt{m} + \sqrt{n} + \sqrt{p} = 2.$$

110. The area of the triangle formed by joining the centres of the escribed circles is

$$\frac{1}{2} S \operatorname{cosec} \frac{\alpha}{2} \operatorname{cosec} \frac{\beta}{2} \operatorname{cosec} \frac{\gamma}{2}.$$

111. Shew that

$$\tan \alpha (\tan 2\alpha)^{\frac{1}{2}} (\tan 4\alpha)^{\frac{1}{4}} = \frac{4 \sin^2 \alpha}{(2 \sin 8\alpha)^{\frac{1}{4}}}.$$

112. Find all the values of x which satisfy the equation

$$\operatorname{vers}^{-1} (1 + x) - \operatorname{vers}^{-1} (1 - x) = \tan^{-1} 2 \sqrt{(1 - x^2)}.$$

Ans. $\pm 1, \frac{1}{2}$.

113. In the three edges of a cube which meet at one angle, three points A, B, C are taken at distances a, b, c from that angle respectively. The area of the triangle ABC is

$$\frac{1}{2} \sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}.$$

114. Determine x from the equation $\tan (\alpha + x) = m \tan x$, and explain the result when $m = 1$.

115. If AD, AE be drawn bisecting the interior and exterior angles at A of a triangle ABC, and r_1, r_2, r_3, r_4 be the radii of the circles circumscribing the triangles ABD, ACD, ABE, ACE, respectively, then $r_1 r_4 = r_2 r_3$.

116. Having given the area, base, and vertical angle of a triangle, find the other angles.

117. Having given two sides of a triangle, and the difference of the angles opposite them, determine the angles.

118. If the angle α be divided into two parts so that their versed sines may be in the ratio $m : n$, one part is

$$2\cot^{-1}\left\{\cot\frac{\alpha}{2}+\sqrt{\frac{n}{m}}\operatorname{cosec}\frac{\alpha}{2}\right\}.$$

119. If r, r' be radii of the circumscribed and inscribed circles of a regular polygon of n sides, shew that

$$r+r'=a\cot\frac{\pi}{2n},$$

where $2a$ is one of the sides.

120. Find the number of acres in a field whose sides are 400, 300, 300, 300 yards respectively, and one angle adjacent to the largest side is a right angle.

Ans. $\frac{125}{242}(5\sqrt{11}+24)$ acres.

121. In a triangle

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=1.$$

122. Shew that $\dfrac{\sin\alpha\pm\sin n\alpha+\sin(2n-1)\alpha}{\cos\alpha\pm\cos n\alpha+\cos(2n-1)\alpha}=\tan n\alpha.$

123. If the sides of a triangle are in arithmetical progression, the perpendicular on the mean side from the opposite angle, and the radius of the escribed circle touching the mean side, are each three times the radius of the inscribed circle.

124. If

$$a\cos^2\theta+b\sin^2\theta=m\cos^2\phi,$$
$$a\sin^2\theta+b\cos^2\theta=n\sin^2\phi,$$
$$m\tan^2\theta=n\tan^2\phi,$$

then

$$(a+b)(m+n)=2mn.$$

125. Solve $8\sin^{-1}x\sin^{-1}2x=3(\sin^{-1}2x)^2-3(\sin^{-1}x)^2.$

Ans. $x=\pm\frac{1}{2}$ or $\pm\frac{\sqrt{14}}{8}$.

126. Solve $8\cos\theta=\dfrac{\sqrt{3}}{\sin\theta}+\dfrac{1}{\cos\theta}.$

Ans. $\theta=\frac{n\pi}{2}+\frac{\pi}{12}$ or $n\pi+\frac{\pi}{3}$.

127. The distance between the centres of two wheels is a, and the sum of their radii is c; shew that the length of a string which wraps round them and crosses between them is

$$2\sqrt{(a^2-c^2)}+c\left(\pi+2\sin^{-1}\frac{c}{a}\right).$$

128. An object a feet high at the top of a tower subtends a given angle α at a distance d from the foot of the tower. The tower's height is

$$-\frac{a}{2}\pm\frac{\sqrt{\{a^2+4d(a\cot\alpha-d)\}}}{2}.$$

Explain the negative result.

129. In a triangle, if a, b, c be in arithmetical progression, then

$$\sin\left(\alpha+\frac{\beta}{2}\right)=2\sin\frac{\beta}{2}.$$

130. Shew that $\cot\gamma=\dfrac{b}{c}\operatorname{cosec}\alpha-\cot\alpha$, and adapt the result to logarithmic computation.

131. Shew that $\tan\theta\tan(60+\theta)\tan(60-\theta)=\tan 3\theta$.

132. If $\dfrac{\sin(x+a)}{\cos(x-a)}=\dfrac{1-m}{1+m}$, then

$$\tan(45^0-x)=m\cot(45^0-a).$$

133. The hypothenuse AB of a right-angled triangle ACB is divided in D so that $AD:BD=CB:CA$. Shew that

$$\tan ACD=\frac{a^2}{b^2},\quad CD=\frac{\sqrt{(a^4+b^4)}}{a+b}.$$

134. Eliminate θ from the equations

$$(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi),$$
$$a\cos 2\phi+b\cos 2\theta=c.$$

Ans. $b^2=a^2+c^2-2ac\cos 2\phi$.

135. If the lines drawn from the angles of a triangle perpendicular to the opposite sides meet in a point whose distances from the three angles are p, q, r, respectively, then the area of the triangle is

$$\frac{1}{4}(pa+qb+rc).$$

Shew also that $a^2p\operatorname{cosec}\alpha+b^2q\operatorname{cosec}\beta+c^2r\operatorname{cosec}\gamma=2abc$.

136. The area of the triangle formed by joining the centres of the escribed circles is $\frac{abc\sqrt{s}}{2\sqrt{\{(s-a)(s-b)(s-c)\}}}$.

137. Shew that

$$\sec^2(\alpha+45)-\sec^2(\alpha-45^0)=4\tan 2\alpha \sec 2\alpha.$$

138. If $\cos(\sigma-2\alpha)+\cos(\sigma-2\beta)=\cos(\sigma-2\gamma)+\cos(\sigma-2\delta)$, where

$$\sigma=\alpha+\beta+\gamma+\delta,$$

then $\tan\alpha\tan\beta=\tan\gamma\tan\delta$.

139. One angle of a triangle is 60^0, and the sides including it are in the ratio 5 : 3. The other angles are

$$\tan^{-1}\frac{3\sqrt{3}}{7} \text{ and } \tan^{-1}5\sqrt{3}.$$

140. If the sides of a triangle ABC be divided into parts at D, E, F, which have to one another the same ratio $n:1$, shew that if the points D, E, F be joined, the triangles ADF, BDE, CEF are all equal, and that

$$\Delta DEF : \Delta ABC = n^2-n+1 : (n+1)^2.$$

141. If $\quad \cos\alpha=\cos\beta\cos\phi=\cos\beta'\cos\phi'$,

and $\sin\alpha=2\sin\frac{\phi}{2}\sin\frac{\phi'}{2}$,

shew that $\quad \tan\frac{\alpha}{2}=\tan\frac{\beta}{2}\tan\frac{\beta'}{2}$.

142. A person walking along a straight road observes that the greatest angle which two objects make with each other is α: from the point where this happens he walks a yards, and the objects there appear in the same straight line making an angle β with the road: the distance between the objects is

$$\frac{2a\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}.$$

143. In any quadrilateral

$$\frac{\tan\alpha+\tan\beta+\tan\gamma+\tan\delta}{\cot\alpha+\cot\beta+\cot\gamma+\cot\delta}=\tan\alpha\tan\beta\tan\gamma\tan\delta.$$

144. Shew that

$$2\tan^{-1}\left[\tan\frac{\alpha}{2}\sqrt{\left\{\tan\left(\frac{\pi}{4}-\beta\right)\right\}}\right]=\cos^{-1}\left(\frac{\cos\alpha+\tan\beta}{1+\cos\alpha\tan\beta}\right).$$

145. A circle is described about a triangle ABC, and a new triangle is formed by joining the points of bisection of the arcs subtended by the sides of ABC; shew that the angles of this triangle are

$$90-\frac{\alpha}{2},\quad 90-\frac{\beta}{2},\quad 90-\frac{\gamma}{2};$$

and the sides are

$$\frac{a}{2\sin\frac{\alpha}{2}},\quad \frac{b}{2\sin\frac{\beta}{2}},\quad \frac{c}{2\sin\frac{\gamma}{2}};$$

and its area : area of triangle $ABC=1 : 8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.

146. Three circles whose radii are a, b, c touch each other externally; prove that the tangents at the points of contact meet in a point whose distance from any one of them is

$$\sqrt{\left(\frac{abc}{a+b+c}\right)}.$$

147. If $a=\operatorname{cosec}\theta\tan^3\theta\,(\operatorname{cosec}^2\theta+1)+\sec\theta$,

and $b=\tan^3\theta\,(\operatorname{cosec}^2\theta+1)-\tan\theta$,

then $a^{\frac{2}{3}}-b^{\frac{2}{3}}=2^{\frac{2}{3}}$.

148. A person wishing to determine the length of an inaccessible wall places himself due south of one end, and due west of the other, at such distances that the angles which the wall subtends at the two positions are each equal to α. If a be the distance between the two positions, the length of the wall is $a\tan\alpha$.

149. A vessel observed another vessel α^0 from the north sailing in a direction parallel to its own. After an hour's sailing its bearing was β^0, and after another hour γ^0. In what direction were the vessels sailing?

Ans. $\frac{\alpha+\gamma}{2}+\tan^{-1}\left\{\tan^2\frac{\gamma-\alpha}{2}\cot\left(\beta-\frac{\alpha+\gamma}{2}\right)\right\}$,

or otherwise, $\tan^{-1}\frac{\tan\beta\,(\tan\alpha+\tan\gamma)-2\tan\alpha\tan\gamma}{2\tan\beta-(\tan\alpha+\tan\gamma)}$.

150. A circle is inscribed in a triangle and tangents drawn parallel to each side and intercepted by the other two. If l, m, n be the lengths of the tangents opposite the sides a, b, c respectively, then

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}=1.$$

151. A circle is inscribed in a triangle; similar triangles are cut off by tangents to the circle, and in these triangles are inscribed other circles; shew that the radius of the first is equal to the sum of the radii of the others.

152. Two squares may be inscribed in a right-angled triangle, one having two sides lying on the sides of the triangle, and the other having one side on the hypothenuse; shew that the latter is always the least.

153. The distance between the centres of the inscribed and circumscribed circles is $\sqrt{(R^2-2Rr)}$.

154. In the triangle ABC, $B'C'$ is drawn through A parallel to BC, and $A'B'$ through C perpendicular to AC, and $A'C'$ through B perpendicular to AB; shew that the area of the triangle $A'B'C'$ is

$$\frac{a^2}{2} \cdot \frac{\cos^2(\beta-\gamma)}{\cos\beta\cos\gamma\sin\alpha}.$$

155. A person walks x yards from A to E along the side AB of the triangle ABC, and observes the angle $AEC=\theta$. Again he walks y yards along BA from B to F and observes $CFB=\theta$. Given $AB=c$, find the other sides of the triangle.

Ans. $\sqrt{\left\{\left(\frac{x+y-c}{2}\sec\theta\right)^2+(c-y)\,x\right\}}$,

and $\sqrt{\left\{\left(\frac{x+y-c}{2}\sec\theta\right)^2+(c-x)\,y\right\}}$.

156. In the hypothenuse AB of a right-angled isosceles triangle ACB, D is taken so that CD is equal to $\frac{5}{7}CB$. Shew that AB is divided in D in the ratio 3 : 4.

157. If

$$c\sin\theta-a\sin(\theta+\phi)=0,$$
$$a\sin\phi-b\sin\theta=0,$$
$$\cos\theta-\cos\phi=2m,$$

then

$$(a-b)\{(a+b)^2-c^2\}+4abmc=0.$$

158. From the summits of two rocks A, B at sea, at a given distance d apart, the dips α, β of the horizon are observed, and it is remarked that the summit of B is in a horizontal line through the summit of A; shew that the radius of the Earth is

$$\frac{d}{\cos^{-1}(\sec\alpha\cos\beta)}.$$

159. Of the three squares which can be described in a triangle so that one side of the square coincides with part of one side of the triangle, the greatest is that which is on the least side.

160. If the perpendiculars from the angular points of triangle ABC meet the sides in D, E, F, and R, R_1 be the radii of the circles described about triangles ABC, DEF, and r_1 the circle inscribed in DEF, prove that $R_1=\frac{1}{2}R$ and $r_1=2R\cos\alpha\cos\beta\cos\gamma$.

161. In a triangle

$$a^2 : b^2 : c^2 = \cot\beta+\cot\gamma : \cot\gamma+\cot\alpha : \cot\alpha+\cot\beta.$$

162. A person walking along a straight road observes the greatest elevation of a tower to be α. From another straight road he observes the greatest elevation of the tower to be β. The distances of the points of observation from the intersection of the two roads are a, b respectively: prove that the height of the tower is

$$\left(\frac{a^2-b^2}{\cot^2\beta-\cot^2\alpha}\right)^{\frac{1}{2}}.$$

163. If the inscribed circle touch the sides of the triangle ABC in points $A'B'C'$, and R_1, R_2, R_3 be the radii of the circles circumscribed about $AB'C'$, $BA'C'$, $CA'B'$ respectively, then $2R_1R_2R_3=r^2R$.

164. If in a triangle ABC circles are described with centres B, C to touch the opposite sides and intersect in A', then

$$\cos BA'C=\frac{\cos\alpha\cos(\beta-\gamma)}{2\sin\beta\sin\gamma}.$$

165. Prove that

$$\tan^{-1}\frac{m+n}{m-n}-\tan^{-1}\frac{m-n}{m+n}=\frac{1}{2}\sin^{-1}\frac{4mn(m^2-n^2)}{(m^2+n^2)^2}.$$

166. Eliminate θ from the equations

$$a\sin 3\theta-b\cos 3\theta=c\sin 2\theta\sqrt{(\cos 2\theta)},$$
$$a\cos 3\theta+b\sin 3\theta=c\cos 2\theta\sqrt{(\cos 2\theta)}.$$

Ans. $(a^2+b^2)^2=c^2(a^2-b^2)$.

167. If $(x^2+y^2)\sin^2\alpha=(y\sin\beta-x\cos\beta)^2$, then $x=y\tan(\beta\pm\alpha)$.

168. If $$x\cos\theta=b,\ x\cos(2\alpha-\theta)=c,$$

then $$x=\frac{1}{2}\{(b+c)^2\sec^2\alpha+(b-c)^2\operatorname{cosec}^2\alpha\}^{\frac{1}{2}}.$$

169. If $\sin\theta=\sin\phi+\sin\psi$, $\cos\phi=\cos\psi+\cos\theta$, $\cot\psi=\cot\theta+\cot\phi$, find θ, ϕ, ψ. *Ans.* $\theta=150^0$, $\phi=90^0$, $\psi=-30^0$.

170. If O be the centre of the circle inscribed in the triangle ABC; D, E, F the points of contact, and ρ_1, ρ_2, ρ_3 the radii of circles inscribed in $AEOF$, $BDOF$. $CDOE$, then

$$r^2(\rho_1+\rho_2+\rho_3)-2r(\rho_1\rho_2+\rho_1\rho_3+\rho_2\rho_3)+2\rho_1\rho_2\rho_3=0.$$

171. Prove that the tangents of the angles of depression of one balloon observed from another at equal intervals of time are in A.P., if the two balloons are under the influence of the same horizontal currents of air, and are moving in straight lines.

172. Solve the equation

$$\cot\theta+\cot\left(\theta+\frac{\pi}{3}\right)+\cot\left(\theta+\frac{2\pi}{3}\right)=3\cot 3a.$$

Ans. $\theta=\frac{n\pi}{3}+a$.

173. Shew that

$$\operatorname{cosec}^2 a+\operatorname{cosec}^2\left(a+\frac{\pi}{3}\right)+\operatorname{cosec}^2\left(a+\frac{2\pi}{3}\right)=9\operatorname{cosec}^2 3a.$$

174. AB is a diameter of a circle produced to any point P, and AC, AD are chords of the circle. If BC, BD, PC, PD are joined, shew that

$$\tan PCB : \tan PDB=\cot BAC : \cot BAD.$$

175. In a triangle, shew that

$$8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}<1,$$

unless $\alpha=\beta=\gamma$. Hence shew that if $R=2r$, the triangle is equilateral.

176. If D, E, F are the feet of the perpendiculars from the angles A, B, C of a triangle on the opposite sides, and D', E', F' the points of contact of the inscribed circle, shew that

$$DD'\sin\alpha+EE'\sin\beta+FF'\sin\gamma=0,$$

if DD', EE', FF' have the same signs as $b-c$, $c-a$, $a-b$ respectively.

177. In a triangle ABC, if D, E, F be the points of contact of the inscribed circle, and AD, BE, CF meet the circle in D', E', F', shew that

$$\frac{AE^2}{AD'^2}+\frac{BF^2}{BE'^2}+\frac{CD^2}{CF'^2}=3+\frac{4}{s}\left\{\frac{r_a^2}{a}+\frac{r_b^2}{b}+\frac{r_c^2}{c}\right\}.$$

178. If $\alpha+\beta+\gamma+\delta=360^0$, and $\tan\alpha$, $\tan\beta$, $\tan\gamma$, $\tan\delta$ be in G.P., then $\tan\alpha\tan\delta=\tan\beta\tan\gamma=1$, and if $\tan\alpha=a$, find the other angles.

Ans. $\tan\beta=\sqrt[3]{a}$, $\tan\gamma=\sqrt[3]{a^{-1}}$, $\tan\delta=a^{-1}$.

179. If
$$\frac{6\sin\theta}{\cos(\phi+\theta)}=\frac{3\sin 2\theta}{\cos(\phi+2\theta)}=\frac{2\sin 3\theta}{\cos(\phi+3\theta)},$$
then $\theta=0$.

180. In a triangle
$$\tan 2\alpha+\tan 2\beta+\tan 2\gamma=-\frac{\sin 4\alpha+\sin 4\beta+\sin 4\gamma}{1+\cos 4\alpha+\cos 4\beta+\cos 4\gamma}.$$

181. In a triangle
$$1+\cos C<2\left(\cos\frac{A-B}{2}+\sin\frac{A+B}{2}\right).$$

182. $ABCD$ is a quadrilateral figure inscribed in a circle whose sides are in G. P. (common ratio r): shew that
$$\tan ABC : \tan BCD = r^2-1 : r^2+1.$$

183. If AD, BC, CF bisect the angles of a triangle ABC, shew that
$$\text{triangle } DEF : \text{triangle } ABC = 2abc : (a+b)(a+c)(b+c).$$

184. If lines be drawn from the angular points A, B, C of a triangle through O the centre of the inscribed circle to meet the circumference of the circumscribed circle in P, Q, R respectively; shew that the product of the radii of the circles circumscribed about BOP, ROA, COQ equals the product of those about BOR, AOQ, COP, and also equals
$$\frac{a^3b^3c^3}{r^2(a+b+c)^4}.$$

185. If p_1, p_2, p_3 be the perpendiculars from the angles on the opposite sides of a triangle, and q_1, q_2, q_3 the perpendiculars from the centre of the circumscribed circle upon the same sides respectively, and r the radius of the inscribed circle, then
$$\frac{r-q_1}{p_1}+\frac{r-q_2}{p_2}+\frac{r-q_3}{p_3}=0.$$

186. A triangle is described about a given circle, and three different similar triangles are escribed about it. Prove that each side of the first triangle is equal to the sum of the corresponding sides of the escribed triangles.

187. If O be the centre of the escribed circle touching the side BC of the triangle ABC, and BO, CO be produced to meet AC, AB produced in F, E, then
$$\frac{OE}{OF}=\frac{\sin\frac{\gamma-\alpha}{2}}{\sin\frac{\beta-\alpha}{2}}.$$

188. If p, q, t be the distances between the centres of the escribed circles of a triangle, then

$$\frac{r_a r_b r_c}{pqt}=\frac{1}{8}\sin\alpha\sin\beta\sin\gamma.$$

Shew also that area of triangle is equal to

$$r_a r_b\sqrt{\frac{t^2}{(r_a+r_b)^2}-1}.$$

189. Prove that $(1+\sec\alpha)(1+\sec 2\alpha)(1+\sec 4\alpha)\ldots$ to n factors is equal to

$$\frac{\tan 2^{n-1}\alpha}{\tan\frac{\alpha}{2}}.$$

190. If A, A' are the peaks of two mountains, and BC a straight horizontal road; shew that the nearer of the peaks will just conceal the other at some point of the road, if $\frac{\sin\alpha}{\sin\beta}=\frac{\sin\alpha'}{\sin\beta'}$, where α is the altitude of A, as seen from any point B of the road, β the angle ABC, and α', β' similar angles for the peak A' as seen from any point B' of the road.

191. ADB is a semicircle; AB $(2a)$ its diameter; CD any radius dividing the semicircle into two sectors in each of which is inscribed a circle: these circles touch AB in M and N. Prove that if $MN=b$, and r, r' be the radii of the two circles,

$$(a-b)^2=(a-2r)(a-2r')=\left(\frac{rr'}{a}\right)^2.$$

192. If

$$a^2=y^2+z^2+2yz\cos\theta,$$
$$b^2=z^2+x^2+2xz\cos\phi,$$
$$c^2=x^2+y^2+2xy\cos\chi,$$

and $\theta+\phi+\chi=\pi$, prove that

$$2yz\sin\theta+2xz\sin\phi+2xy\sin\chi=\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}.$$

193. If the sides of a triangle ABC are divided in points A', B', C' so that

$$BA':A'C=CB':B'A=AC':C'B=m:n;\text{ where } m+n=1,$$

and if AA', BB', CC' be joined and a triangle formed whose sides are equal to these three lines, then its area equals $(1-mn)$ times the area of ABC.

194. A man walks directly towards an object A until two other objects B, C subtend the greatest angle in his path, when he observes that A and B appear under the same angle (α) as B and C. He then walks directly towards C until A and B subtend the greatest angle in his new path, when he once more observes that A and B appear under the same angle (β) as B and C. If the length of his walk between the two observations be d, shew that $2\beta=90^\circ+\alpha$, and find the distances AB, BC, AC.

Ans. $AB=d\sin\alpha\sec\beta$, $BC=d\tan\beta$,

$$AC=d\sqrt{1+4\sin\alpha+12\sin^2\alpha+8\sin^3\alpha}.$$

195. The sun's rays supposed parallel, cast a shadow throughout the day on a horizontal table of a sphere resting on the table. The sun's altitude is α when it is S.S.E., and β when it is W.S.W.: prove that if D be the distance between the two points of the shadows farthest from, and d the distance of the two points nearest to, the point of contact of the sphere,

$$d = D \tan\frac{\alpha}{2}\tan\frac{\beta}{2}.$$

196. Solve the equation

$$\cos\theta + \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} - \frac{2\sin^2\frac{\theta}{2}}{\cos\theta} = \cos\alpha + \frac{1}{2}\operatorname{cosec}^2\frac{\alpha}{2} - \frac{2\sin^2\alpha}{\cos\alpha}.$$

$$\textit{Ans.}\quad \sin\frac{\alpha+\theta}{2} = 0,\quad \sin\frac{\alpha-\theta}{2} = 0,\quad \cos\theta = \frac{1}{1-\cos\alpha} \text{ or } \frac{\cos\alpha - 1}{\cos\alpha}.$$

197. Two circles (radii R, r) touch each other internally. A tangent to the smaller circle is a chord of the larger, and is inclined at an angle θ to their common diameter, shew that its length is

$$2\sqrt{(R-r)(1+\sin\theta)\{R(1-\sin\theta)+r(1+\sin\theta)\}}.$$

198. O is the centre of the inscribed circle in the triangle ABC; P, Q, R are the centres of the circles inscribed in BOC, COA, AOB respectively: prove that the area of the triangle PQR : the area of ABC ::

$$\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{C}{4}\left(\tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}+3\right) : \sqrt{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

199. O and E are the centres of the circles inscribed in and circumscribed about the triangle ABC, and D is the intersection of the perpendiculars from A, B, C on the opposite sides; prove that the area of the triangle

$$DEO : \text{area of } ABC :: \sin\frac{C-B}{2}\sin\frac{A-C}{2}\sin\frac{B-A}{2} : \sin A\sin B\sin C.$$

200. The inscribed circle of a triangle passes through the centre of the circumscribing circle, and also through the point of intersection of the perpendiculars from the angles on the opposite sides, prove that the angles of the triangle are

$$\frac{\pi}{3},\quad \frac{\pi}{3}+\cos^{-1}\left(\sqrt{2}-\frac{1}{2}\right),\quad \frac{\pi}{3}-\cos^{-1}\left(\sqrt{2}-\frac{1}{2}\right).$$

201. The tops of three mountains B, C, D are in the same plane with the station A. D is invisible from A, but the angles of elevation of B and C are observed to be θ and ϕ, and the angle BAC to be α. From B the angles ABD, CBD are observed to be β and α respectively, and from C the angle ACD is found to be β; prove that D is at a height above A equal to

$$\frac{a\sin\beta(2\cos\alpha\sin\phi - \sin\theta)}{\sin(2\alpha+\beta)},$$

where $AB = a$.

202. In a triangle if

$$y\sin^2\alpha+x\sin^2\beta=z\sin^2\beta+y\sin^2\gamma=x\sin^2\gamma+z\sin^2\alpha,$$

then $$x:y:z=\sin 2\alpha:\sin 2\beta:\sin 2\gamma.$$

203. In a triangle

$$\sin\left(\gamma+\frac{\alpha}{2}\right)+\sin\left(\beta+\frac{\gamma}{2}\right)+\sin\left(\alpha+\frac{\beta}{2}\right)+1=4\cos\frac{\alpha-\beta}{4}\cos\frac{\alpha-\gamma}{4}\cos\frac{\beta-\gamma}{4}.$$

204. In a triangle

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}$$

cannot be less than $\frac{3}{4}$.

205. If
$$x+y\cos\gamma+z\cos\beta=\cos(\sigma-\alpha),$$
$$y+z\cos\alpha+x\cos\gamma=\cos(\sigma-\beta),$$
$$z+x\cos\beta+y\cos\alpha=\cos(\sigma-\gamma),$$

then $$\frac{x}{\sin\alpha}=\frac{y}{\sin\beta}=\frac{z}{\sin\gamma}=\frac{1}{2\sin\sigma}$$

where $2\sigma=\alpha+\beta+\gamma$.

206. If $$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$

then $(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)^2+4\{\sin^2(\beta-\gamma)+\sin^2(\alpha-\beta)+\sin^2(\alpha-\gamma)\}=8.$

207. Prove that the area of any triangle is a mean proportional between the areas of two triangles formed, the one by joining the points of contact of any one of the four circles touching the sides, and the other by joining the centres of the other three.

208. A plane polygon, whose sides are $a_1, a_2, a_3, \ldots$ and area A, is divided into triangles by joining its angular points with a point at which the sides subtend angles $\theta_1, \theta_2 \ldots$ If the centres of the circumscribing circles of these triangles be joined in order, the area of the polygon thus formed is

$$\frac{1}{2}A-\frac{1}{8}(a_1^2\cot\theta_1+a_2^2\cot\theta_2+\ldots).$$

209. The angles subtended by one tower at the base and summit of another tower of height a on the same horizontal plane, are respectively α and β. Prove that the height of the first tower is given by the equation

$$x^2\cos\alpha\sin(\beta-\alpha)-ax\sin^2\alpha\sin\beta+a^2\sin^2\alpha\sin\beta=0.$$

210. Two equal circles roll on the circumference of another circle, the one inside and the other outside: if when they have completed the

same number of circuits, they have made complete revolutions which are in number as 7 : 1, compare the radii of the circles. *Ans.* 3 : 4.

When the exterior circle has performed half a circuit, where must the interior one be so as to have revolved as much?

It has performed 3½ circuits.

211. If the sides a, b, c of a triangle be increased in the ratios $l : 1$, $m : 1$, $n : 1$, and the angle c be thereby halved and the sum of the other two doubled, then

$$(la^2+mb^2)(l+m)=c^2(lm+n^2).$$

212. From the extremity B of the radius AB of a circle a straight line BC is placed in the circle $=\frac{1}{2}$ radius, from C another straight line is drawn through the middle point of AB to meet the circumference in F. Shew that triangle BFA is double the triangle BCA. Also if $FAB=\phi$, $CAB=\theta$, then

$$\operatorname{cosec}\phi+\cot\theta+3\cot(\phi+\theta)=0.$$

213. A rod is moved towards the eye of an observer so that its middle point and upper extremity are always viewed in the same directions: if at two positions when its upper and lower halves subtend angles at the eye which are as $1:1$, $1:2$, the distances of its middle point from the eye are $a\sqrt{5}$, a: find the length of the rod. *Ans.* $2a\sqrt{5}$.

214. At the extremity of an arc (AB) of a circle (radius r) subtending an angle θ, a tangent BD is drawn in the direction of angular movement, whose length is $r\tan\theta$. If AC is the diameter through A, and DC be joined making angle $DCA=\phi$, then if when θ is doubled ϕ becomes ϕ', shew that

$$\frac{\tan\phi}{\tan\phi'}=\frac{\cos 3\theta}{\cos\theta+\cos 2\theta}\cdot\frac{\cos\theta}{\cos 2\theta}.$$

215. A trefoil consisting of three equal semicircles is inscribed symmetrically within another such trefoil: show that their areas are in the same ratio as the radii of the circles inscribed in the smaller and circumscribing the greater trefoil.

216. A person wishing to calculate the radius of a railway curve, measures a distance a from one of the telegraph posts at right angles to the curve at that point: he then observes that the pth and qth posts from the one, from which he made his measurement, are in a straight line with him: the distance between two consecutive posts, measured along the curve, is a, shew that the radius is formed from the equation

$$\cos(p+q)\frac{a}{2r}=\frac{r}{r+a}\cos\frac{(q-p)a}{2r}.$$

EXAMINATION PAPERS.

PREVIOUS EXAMINATION. *December*, 1866.

1. WHAT is the circular measure of an angle? The ratio of the circumference of a circle to its diameter is 3·14159; express in degrees (to four places of decimals) the angle whose circular measure is unity.

2. Define the sine, cosine, and cotangent of an angle. Trace the changes in sign and magnitude of the cosine as the angle increases from 0^0 to 360^0.

3. Investigate a general expression for all angles which have a given tangent.

4. Express the cosine of an angle in terms of the tangent only, and of the cosecant only.

Solve the equation, $4\sec^2\theta - 3\tan^2\theta = 1$.

5. Shew that $\sin(A+B) = \sin A \cos B + \cos A \sin B$, where A and B are two angles whose sum is less than 90^0.

Given $\sin 45^0 = \frac{1}{\sqrt{2}}$, $\sin 30^0 = \frac{1}{2}$, find $\sin 15^0$.

6. Shew that in every triangle any side is equal to the sum of the products obtained by multiplying each of the other sides into the cosine of the angle between it and the first side.

April, 1867.

1. DESCRIBE the different ways of measuring angles.

Write down to five places of decimals the ratio which the circumference of a circle bears to its diameter.

2. An arc of a circle whose radius is 7 inches, subtends an angle of $15^0.39'.7''$: what angle will an arc of the same length subtend in a circle whose radius is 2 inches?

3. There is an angle whose tangent is twice its sine: find the length of the arc subtending it in terms of the radius.

4. Prove the formula

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

5. Shew that the difference of cosines of any two angles is equal to twice the product of the sine of half the sum of the angles, and the sine of half the difference.

6. If ABC be a triangle, prove that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

7. Solve the equation

$$2\sin^2 3\theta + \sin^2 6\theta = 2.$$

8. The town C is halfway between the towns D and E: and the towns C, E and F are equidistant from each other. Compare the distance of D from F with its distance from E.

December, 1868.

1. DEFINE an angle. How are angles measured? Compare the measures of any angles referred to the three usual unit angles. What is the circular measure of an English minute?

Express $\frac{4}{5}$ of two right angles in each of the French and circular systems. If the unit of measurement be 5^0, what is the measure of $22\frac{1}{3}^0$?

2. Define the tangent of an angle; draw the figure of an angle whose tangent is 2. What is the cosecant of this angle? Can more than one such angle be drawn? If the tangent be negative, what inference may be drawn as to the magnitude of the angle?

3. What is a supplement? What are the supplements of 25^0? of 205^0? of $-\frac{3\pi}{4}$? State and prove the relations subsisting between the secants of A, (90^0+A), and of θ, $\frac{\pi}{2}-\theta$.

4. Prove Geometrically that

$$\cos 24^0=\cos 60^0 \cos 36^0+\sin 60^0 \sin 36^0.$$

Given $\cos 36^0=\frac{\sqrt{5}+1}{4}$, $\cos 60^0=\frac{1}{2}$; find $\cos 24^0$ to three places of decimals.

5. Establish the following formulæ:

(1) $\sec^2 x-\tan^2 x=1.$

(2) $\cos 4x=\cos^2 2x-\sin^2 2x.$

(3) $\sin 2y+\sin 2z=2\sin(y+z)\cos(y-z).$

6. Obtain an expression for the area of a triangle.

What is the area of a triangle whose base is 4 ft., and altitude $1\frac{1}{4}$ yds.? What of a triangle whose sides are 5, 6, 5 inches respectively?

7. A target is 6 feet in height and 4 feet in breadth; find the tangents of the angles which its 4 edges subtend at a point 100 yards straight in front of its top left-hand corner.

March, 1869.

1. AN angle referred to different units has measures in the ratio 8 to 5, one unit is 2^0, what is the other? Express each unit in terms of the other.

2. Trace the changes in sign of $\cos A$ as A changes from 45^0 to 315^0. Illustrate by a figure.

3. Shew that $\sin A$ is numerically greatest when $\cos A$ is numerically least, and vice versâ. When is $\cos^4 A-\sin^4 A$ greatest?—and when least?

4. What relation have the cosine, cosecant, and cotangent of an angle to the sine, secant, tangent respectively?

Find the difference between the supplement of the complement of an angle, and the complement of its supplement.

5. Prove that $\tan 45^0=1$, and $\tan 60^0=\sqrt{3}$. Hence find $\tan 15^0$ to 3 places of decimals.

6. Prove that

$$(1) \quad \sec^2 \alpha+\operatorname{cosec}^2\alpha=\sec^2 \alpha \operatorname{cosec}^2 \alpha.$$

$$(2) \quad \cot^2 \alpha-\tan^2\alpha=\frac{4\cos 2\alpha}{\sin^2 2\alpha}.$$

7. In any triangle obtain an expression for the cosine of each angle in terms of the sides, e.g. let the sides be 5, 7, 8. Find also the radius of the circle which circumscribes this triangle.

8. From G the centre of gravity of an equilateral triangle GB is drawn at right angles to the plane of the triangle and equal to a side; determine the angle between the lines joining B to any two corners of the triangle.

December, 1871.

1. WHAT are the three different methods employed in measuring angles, and what are their respective units?

Find an expression for an angle of a regular hexagon referred to an angle of a regular pentagon as unit.

2. Define $\sin A$, $\cot A$, $\sec A$: and express any two in terms of the third.

If $\tan A=\frac{1}{\sqrt{8}}$, find the value of $\sin A$.

3. Find an expression for $\cos(A-B)$ in terms of the sines and cosines of A and B, when A and B are each less than 90^0, and A greater than B.

Shew that the sum of the cosines of two angles is equal to twice the cosine of half the sum of the angles multiplied by the cosine of half their difference.

4. Find the value of $\sin 30^0$, and of $\cos 45^0$.

5. Prove the formulæ:

$$(1) \quad \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

$$(2) \quad 2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}.$$

Given $\sin 30^0$, find $\sin 15^0$.

6. Find a general expression for all angles that have a given cosine.

If $\sin\theta+\cos\theta=\sqrt{2}$, find the value of θ.

7. In any triangle, shew that the sides are proportional to the sines of the opposite angles.

Hence deduce an expression for the cosine of an angle of a triangle in terms of the sides.

8. The sides of a triangle are 3 in., 5 in., 7 in. Find the greatest angle.

S. JOHN'S COLLEGE. *May*, 1868.

1. **Define** the Trigonometrical Ratios. Shew that

$$\sec^2 A = 1 + \tan^2 A.$$

Find the least value of $a^2 \sec^2 x + b^2 \cos^2 x$, where a and b are constant quantities.

2. Investigate an expression for all the angles which have a given cosine. Write down the general value of x which satisfies the equation

$$\sin^2 x = \sin^2 \alpha.$$

3. Shew that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

and

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

If $A + B + C = 180^\circ$, shew that

$$\sin^2 A \sin 2A + \sin^2 B \sin 2B + \sin^2 C \sin 2C - \sin 2A \sin^2 B - \sin 2B \sin^2 A$$
$$= \sin 2A \sin 2B \sin 2C.$$

4. If θ be the circular measure of a positive angle less than a right angle θ is greater than $\sin\theta$ and less than $\tan\theta$.

Find the limit when θ is indefinitely diminished

$$\text{of } \frac{\sin a\theta}{\sin b\theta}, \text{ and } \frac{\operatorname{vers} a\theta}{\operatorname{vers} b\theta}.$$

7. Investigate an expression for the area of a triangle in terms of the sides.

The sides a, b, c of a triangle are in arithmetical progression: shew that the area is to that of an equilateral triangle having the same perimeter as

$$\sqrt{1 - \frac{4(a-b)^2}{b^2}}$$

is to unity.

8. Shew that in any triangle,

$$a \sin \frac{1}{2}(B-C) = (b-c)\cos\frac{A}{2},$$

and

$$a \cos \frac{1}{2}(B-C) = (b+c)\sin\frac{A}{2}.$$

If $b = 14$, $c = 11$, $A = 60^\circ$, find B and C, having given

$$\log 2, \log 3, \text{ and } \log \tan 11^\circ.\ 44'.\ 29''.\ 5.$$

9. Explain the notation used for inverse trigonometrical functions.

Find the value of $\cos 4(\tan^{-1} a)$ in terms of a.

Solve the equation

$$\tan^{-1} x + \frac{1}{2}\sec^{-1} 5x = \frac{\pi}{4}.$$

May, 1869.

1. EXPLAIN what is meant by the circular measure of an angle. Why is it necessary to shew that the angle subtended at the centre of a circle by an arc equal to the radius is an invariable angle? Shew how to connect the circular measure of an angle with the measure of the same angle in degrees.

The difference of two angles is 1°; the circular measure of their sum is 1: find the angles.

2. Define the Trigonometrical Ratios. Shew that $\sin^2 A+\cos^2 A=1$. Find the value of θ from the equation

$$\sin^2\theta-2\cos\theta+\frac{1}{4}=0.$$

Investigate the conditions which must hold in order that the equation

$$\sin^2\theta+b\sin\theta+c=0$$

may have *two* admissible roots, or may have *one*.

3. Investigate the expressions for $\sin(A+B)$ and $\cos(A+B)$ in terms of the sines and cosines of A and B.

Solve the following equations:

$$(1)\quad \sin 3\theta=8\sin^3\theta.$$

$$(2)\quad \sin 5\theta=16\sin^5\theta.$$

4. Find the sine and cosine of an angle of 18°.

Find the values of θ between 0 and $\frac{\pi}{2}$ which satisfy the equation

$$\sin^2\theta+\cos^2 2\theta=\frac{3}{4}.$$

5. Define a logarithm. Explain the advantage of the common system of logarithms.

Find approximately the value of x from the equation

$$\left(\frac{10}{3}\right)^{x+2}=9^{2x-1};$$

having given $\log 3=\cdot4771213$.

7. Express the cosine of an angle of a triangle in terms of the sides.

A person on the slope of a hill observes the angles of elevation α and β of two objects on the hill; and also the angle γ which they subtend at his position: if θ be the inclination of the hill to the horizon, shew that

$$\sin^2\theta\sin^2\gamma=\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos\gamma.$$

8. Shew how to solve a triangle, having given two sides and the included angle.

If $a=210$, $b=110$, $C=34^\circ.\,42'.\,30''$, find the other angles, having given

$$\log 2=\cdot3010300,$$

$$L\cot 17^\circ.\,21'.\,15''=10\cdot5051500.$$

9. Find the area of a circle, and of a sector of a circle.

CAMBRIDGE LOCAL EXAMINATIONS.

JUNIOR STUDENTS, 1880.

1. The sine of an angle is $\frac{11}{61}$; find its secant and tangent.

Draw figures to represent the four least positive angles, of which the cosine is double the sine.

2. Prove that $\tan A = \cot (90^\circ - A)$.

Express $\sin 2300^\circ$, $\tan 800^\circ$ in terms of trigonometrical ratios of angles less than 45°.

3. Prove geometrically or otherwise that

(i) $\sin (A+B) + \sin (A-B) = 2 \sin A \cos B$.

(ii) $\tan (C+D) = \dfrac{\tan C + \tan D}{1 - \tan C \tan D}$.

(iii) $\sin 2E = 2 \sin E \cos E$.

4. Prove

(i) $(1 - 2\cos^2 A)(\tan A + \cot A) = (\sin A - \cos A)(\sec A + \operatorname{cosec} A)$.

(ii) $\sin 3A = \sin A \, (2 \cos 2A - 1) \tan (60^\circ + A) \tan (60^\circ - A)$.

5. Solve the equation $\tan 5\theta = \tan \theta$.

6. Explain the use of logarithms in facilitating the processes of multiplication and division.

Find the following logarithms: $\log_2 1024$, $\log_{\frac{1}{5}} 125$, $\log_{\cdot\dot{6}} \cdot\dot{4}$; and the characteristics of the following: $\log_3 250$, $\log_{\frac{1}{3}} 87$, $\log_{10} \cdot 00\dot{1}$.

7. Prove that in any triangle of which A, B, C are the angles, and a, b, c the sides,

$$c = a \cos B + b \cos A.$$

Express the distance of the middle point of a side of a triangle from the perpendicular upon that side from the opposite angle in terms of the sides of the triangle.

8. Find the number of cubic inches of metal in a hollow right circular iron pipe, height 10 ft. 4 in., outer radius 9 in. and thickness $1\frac{3}{4}$ in.: obtain your result accurate to a cubic inch, assuming $\pi = 3{\cdot}1416$.

JUNIOR STUDENTS, 1882.

1. A certain small angle was found by a Frenchman to measure 27·2 French seconds: an Englishman found it to measure 8·82 English seconds. Do the two measures agree? If not, find their difference in English seconds to three decimal places.

2. State carefully what you mean by the sine and the cosine of an obtuse angle.

Write down the values of $\sin 90^0$, $\sin 270^0$, $\cos 180^0$, $\cos 360^0$, $\tan 450^0$.

3. What is the *base* of a system of logarithms? Prove that $\log(\sqrt[n]{x}) = \frac{1}{n}\log x$.

Find the seventh root of ·004, having given that

$$\log 2 = \cdot 30103 \text{ and } \log 45439 = 4\cdot 65743.$$

4. Prove the following identities:

(i) $\sin^2 A + \sin^2 B = 1 - \cos(A - B)\cos(A + B)$;

(ii) $\cos A + \cos(120^0 - A) + \cos(120^0 + A) = 0$.

5. In any triangle the tangent of half the difference of two angles is to the tangent of half their sum as the difference of the sides respectively opposite to the angles is to the sum of these sides. Prove this.

Find the angles B and C in the triangle ABC from these data:

$$A = 60^0,\ b = 17,\ c = 7,\ \log 2 = \cdot 30103,\ \log 3 = \cdot 47712,$$

$$L\tan 35^0\,49' = 9\cdot 85833,\ L\tan 35^0\,49'\,10'' = 9\cdot 85838.$$

6. A lighthouse facing N. sends out a fan-shaped beam extending from N.E. to N.W. A steamer sailing due W. first sees the light when 5 miles away from the lighthouse and continues to see it for $30\sqrt{2}$ minutes. What is the speed of the steamer?

7. Find the area of the entire surface of a right cone whose height is 4 feet and diameter of base also 4 feet. Give the answer in square feet correct to two places of decimals.

SENIOR STUDENTS, 1880.

1. DEFINE the unit of circular measure.

A line turning about a point makes three revolutions and a sixth, what are the measures of the angle it has turned through in circular measure and in degrees?

2. Define the sine and cosine of an angle.

Prove that $\sin^2 A + \cos^2 A = 1$, and that $\sin(90^0 - A) = \cos A$, and hence deduce the value of $\sin 45^0$.

3. Prove that

(1) $\cos(A + B) = \cos A \cos B - \sin A \sin B$,

(2) $$4\cos(A + B + 45^0)\cos(A + B - 45^0)\cos(A - B) = \cos(A + 3B) + \cos(3A + B).$$

4. Solve the equation

$$2\sin^2\theta+\sin 2\theta+\cos 2\theta=0.$$

5. Express the sine of half an angle in terms of the sine of the angle, and thence find the value of $\sin 67^\circ.\ 30'$.

6. Investigate formulæ for solving a triangle when two sides and the included angle are given.

Ex. Having given $A=60^\circ$, $b=14$ ft., $c=11$ ft.,

$$\log 4{\cdot}32={\cdot}6354837,\quad L\tan 11^\circ.\ 44'=9{\cdot}3174299,$$

$$L\tan 11^\circ.\ 45'=9{\cdot}3180640,$$

solve the triangle.

7. Find the radius of the circle circumscribing a given triangle.

From the angular points of a triangle ABC perpendiculars are drawn to the opposite sides meeting them in D, E, F. Find the sides and angles of the triangle DEF, and thence the radius of the circle circumscribing it.

SENIOR STUDENTS, 1881.

1. ESTABLISH a method for finding the number of degrees in an angle whose circular measure is given.

If the sum of the numbers expressing the measures of an angle in degrees and in circular measure be 90, determine the angle.

2. Prove that $\sec^2 A=1+\tan^2 A$, and that

$$\sin(90^\circ+A)=\cos A.$$

If $\tan A=\frac{a}{b}$, find the value of $\cos A$ and of $\tan(45^\circ+A)$.

3. Prove that

$$(1)\quad \cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$

$$(2)\quad \cos(A-2B)-\cos(A-B)+\cos A-\cos(A+B)+\cos(A+2B)$$

$$=\cos A\,\frac{\cos\frac{5}{2}B}{\cos\frac{1}{2}B}=4\cos A\,(\cos B-\cos 36^\circ)(\cos B+\cos 72^\circ).$$

4. Solve the equation

$$\cos 2\theta+2\sin^2 2\theta=1.$$

5. Shew how to find $\cos A$ in terms of $\sin 2A$.

Hence find the values of $\cos 15^\circ$, $\cos 75^\circ$, $\cos 195^\circ$, $\cos 255^\circ$.

6. Prove that in any triangle

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

The sides of a triangle are 4, 5, 6 feet respectively, determine the least angle, having given

$$\log 7=\cdot 845098$$

$$L\tan 20^\circ.\,42'=9\cdot 577341$$

$$L\tan 20^\circ.\,43'=9\cdot 577723.$$

SENIOR STUDENTS, 1882.

1. **Define** the unit of circular measure, and explain what is meant by the circular measure of an angle. Assuming that $\pi=3\cdot 1416$, find the number of degrees in this unit.

2. Define the trigonometrical ratios of an angle. Compare them with the corresponding ratios of its supplement.

If $\cos A=\frac{m^2-n^2}{m^2+n^2}$, find $\sin A$ and $\tan A$.

3. Prove that

(i) $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

(ii) $(\sin 2A-\sin 2B)\tan(A+B)=2(\sin^2 A-\sin^2 B)$.

4. Find an expression for all the angles which have a given tangent.

Find all the values of θ which satisfy the equation:

$$(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta.$$

5. In any triangle the sides are proportional to the sines of the opposite angles.

If points D, E, F be taken in the sides of a triangle ABC so that

$$\frac{BD}{BC}=\frac{CE}{CA}=\frac{AF}{AB}=\frac{p}{q},$$

prove that

$$\frac{\cot DAC+\cot EBA+\cot FCB}{\cot A+\cot B+\cot C}=\frac{q+p}{q-p}.$$

6. Prove that in any triangle

$$\tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

If $a=27$, $b=23$, $C=40^\circ.30'$; find A and B having given

$L\cot 22^\circ.15'=10\cdot 3881591$, $L\tan 11^\circ.\,3'=9\cdot 2906713$,

$\log 2=\cdot 3010300$, $L\tan 11^\circ.\,4'=9\cdot 2913424$.

ENTRANCE EXAMINATION, SANDHURST. 1881.

1. WHAT is meant by the unit of circular measure?

Prove the formula $\theta = \frac{\text{arc}}{\text{radius}}$.

Find the length of that part of a circular railway-curve which subtends an angle of $22\frac{1}{2}^0$ to a radius of a mile. $(\pi = 3{\cdot}1416.)$

2. Prove from a figure that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B,$$

when A lies between 315^0 and 360^0, and $A - B$ between 180^0 and 225^0.

3. Prove that

$$2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}.$$

Shew, *a priori*, the reason of the four different values of $\sin \frac{A}{2}$ found from $\sin A$.

4. Prove that

(1) $\sec^4 \theta + \tan^4 \theta = 1 + 2 \sec^2 \theta \tan^2 \theta.$

(2) $\sin^{-1} \left(\frac{x - a + b}{2b} \right)^{\frac{1}{2}} = \frac{1}{2} \cos^{-1} \frac{a - x}{b}.$

(3) $\tan 7^0.\, 30' = (\cot 30^0 - \operatorname{cosec} 45^0)(\sec 45^0 - 1).$

5. Shew that in any triangle ABC

$$(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c,$$

and if r, R, r_a, r_b, r_c are the radii of the circles inscribed in, circumscribed about, and escribed to the triangle ABC,

$$\frac{1}{r_a - r} + \frac{1}{r_b + r_c} = \frac{4R}{a^2}.$$

6. The three sides BC, CA, AB of a triangle are as 4 : 5 : 6. Find the angle B.

$$\log 2 = {\cdot}3010300,$$

$$L \cos 27^0.\, 53' = 9{\cdot}9464040, \qquad L \cos 27^0.\, 54' = 9{\cdot}9463371.$$

7. The angular elevation of a steeple at a place due south of it is 45^0, and at another place due west of the former station the elevation is 15^0: shew that the height of the steeple is $\frac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$, a being the distance between the places.

ENTRANCE EXAMINATION, SANDHURST. 1882.

1. DISTINGUISH between the circular measure of an angle, and its measure in degrees; and prove that to turn circular measure into seconds we must multiply by 206265, and to turn seconds into circular measure we must multiply by ·000004848.

2. Compare the Trigonometrical Ratios of any angle and its supplement: and determine the Trigonometrical Ratios of 495^0.

Find all the angles between 0^0 and 500^0 which satisfy the equation

$$\sin^2\theta=\frac{3}{4}.$$

3. Find geometrically an expression for the cosine of the difference of two angles in terms of the trigonometrical ratios of those angles.

Find all the values of x, which satisfy the equation

$$\cos x\cos 3x=\cos 2x\cos 6x.$$

Solve the equation

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}x=\frac{\pi}{4}.$$

4. If r, r_1, r_2, r_3 be the radii of the inscribed and escribed circles of a triangle, and x, y, z the perpendiculars from the angles on the opposite sides, prove

$$(1)\quad xyz=\frac{(a+b+c)^3}{abc}r^3.$$

$$(2)\quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}=\frac{r}{\sqrt{(r_2-r)(r_3-r)}}.$$

5. In a triangle, prove that

$$\tan\left(\frac{A}{2}+B\right)=\frac{c+b}{c-b}\tan\frac{A}{2};$$

and if $3c=7b$, and $A=6^0.\,37'.\,24''$, find the other angles.

Given $\log 2=\cdot30103$, $\quad L\tan 3^0.\,18'.\,42''=8\cdot7624069$,

$L\tan 8^0.\,13'.\,50''=9\cdot1603083$, diff. for $10''=1486$.

6. In a plane triangle, prove that

1. $\tan A\tan B\tan C=\tan A+\tan B+\tan C$,

2. $a\sin A+b\sin B+c\sin C=2(\alpha\cos A+\beta\cos B+\gamma\cos C)$,

where a, b, c are the sides, and α, β, γ the perpendiculars let fall on them from the opposite angles respectively.

7. Prove that the area of a triangle

$$=\frac{a^2}{4}\sin 2B+\frac{b^2}{4}\sin 2A;$$

and if R, r are the radii of the circumscribing and inscribed circles

$$Rr=\frac{abc}{4(a+b+c)}.$$

8. Given $\log 1\frac{1}{5}=\cdot 0791812$, and $\log 2\frac{2}{5}=\cdot 3802112$, find the value of

$$\sqrt[5]{(3.6)^3}\times\sqrt[4]{\tfrac{1}{25}}\div\sqrt[3]{8\tfrac{49}{72}},$$

the mantissæ for 46929 and 46930 being 6714413 and 6714506.

9. A measured line is drawn from a point on a horizontal plane in a direction at right angles to the line joining that point to the base of a tower standing on the plane. The angles of elevation of the tower from the two ends of the measured line are 30° and 18°. Find the height of the tower in terms of l the length of the measured line.

ENTRANCE EXAMINATION, WOOLWICH. *June*, 1882.

1. PROVE that the angle subtended at the centre of a circle by an arc equal in length to its radius is an invariable angle.

One angle of a triangle is 45°, and the circular measure of another is $1\frac{1}{2}$. Find the third, both in degrees and in circular measure.

2. Define the secant of an angle, and shew how your definition applies to angles between 180° and 270°.

If $\sec A=-2$, what two values between 0° and 360° may A have?

3. Obtain a formula embracing all the angles which have a given tangent.

Determine all the values of θ which satisfy the equation:

$$\sqrt{3}\tan^2\theta+1=(1+\sqrt{3})\tan\theta.$$

4. Find an expression for $\tan 3A$ in terms of $\tan A$. Shew also that

$$\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A.$$

5. Prove that

$$\sin 18^\circ=\frac{\sqrt{5}-1}{4};$$

and that $$\sin^2 30^\circ=\sin 18^\circ\sin 54^\circ.$$

Shew that in any circle the chord of an arc of 108° is equal to the sum of the chords of arcs of 36° and 60°.

6. Demonstrate the identities

(1) $\dfrac{(\text{cosec}\, A + \sec A)^2}{\text{cosec}^2 A + \sec^2 A} = 1 + \sin 2A,$

(2) $\sin 3A = 4 \sin A \sin (60 + A) \sin (60 - A),$

(3) $4 (\cot^{-1} 3 + \text{cosec}^{-1} \sqrt{5}) = \pi.$

7. What are the advantages gained by the use of logarithms calculated to the base 10?

If $\log_{10} 2 = \cdot 30103$, find the logarithms of 5, $\dfrac{1}{125}$, $4 \sqrt{\cdot 005}$ to the base 10.

8. Prove that in any triangle

(1) $2bc \cos A = b^2 + c^2 - a^2.$

(2) $\dfrac{1 + \cos (A - B) \cos C}{1 + \cos (A - C) \cos B} = \dfrac{a^2 + b^2}{a^2 + c^2}.$

9. If r_1 be the radius of a circle touching the side a of a triangle, and the other two sides produced, shew that

$$r_1 \cos \frac{A}{2} = a \cos \frac{B}{2} \cos \frac{C}{2}.$$

If a be the side of a regular polygon of n sides, and R, r the radii respectively of its circumscribed and inscribed circles, prove that

$$R + r = \frac{a}{2} \cot \frac{\pi}{2n}.$$

10. Two sides of a triangle, which are respectively 250 and 200 yards long, contain an angle of 54°. 36′. 24″.

Find the two other angles, having given

$$L \cot 27^{\circ}.\, 18' = 10\cdot 2872338, \text{ diff. for } 1' = 3100,$$
$$L \tan 12^{\circ}.\, 8'.\, 50'' = 9\cdot 3329292 : \log 3 = \cdot 4771213.$$

11. The eye of a soldier in a straight trench of uniform depth is 2 feet above a level plain on which he sees two men standing in the same straight line as the trench; the parts of their bodies above the level of his eye subtending at it the angles $\tan^{-1} \cdot 0041\dot{6}$, and $\tan^{-1} \cdot 00\dot{4}$. On walking 200 feet towards them in the trench, he notices that the height of one exactly hides that of the other; and on approaching 596 feet 8 inches closer still he finds that the portion of the height of the nearer above the level of his eye subtends at it 45°. Find the heights of the men.

PREVIOUS EXAMINATION. *June*, 1880.

1. DISTINGUISH between Euclid's definition of an angle and the trigonometrical definition.

What angle does the minute-hand of a clock describe between half-past four and a quarter past six?

2. Define the cosine of an angle, and trace the change in the value of the cosine as the angle increases from 180^0 to 360^0.

3. Express the sine and the cosine of an angle in terms of the tangent.

The angle A is greater than 180^0 but less than 270^0, and $\tan A = \frac{1}{2}$: find $\sin A$.

4. Find $\cos 30^0$ and $\cot 135^0$.

5. Prove geometrically $\cos(A-B) = \cos A \cos B + \sin A \sin B$, A and B being both positive angles less than 90^0.

Shew that

(1) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$,

(2) $\sin 2A = \dfrac{2 \cot A}{1 + \cot^2 A}$.

6. Shew that if $A + B + C = 90^0$,

$$\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C.$$

7. Find an expression for all the values of θ for which $\sin\theta + \sin 2\theta = 0$.

8. Express the cosine of an angle of a triangle in terms of the sides.

If in a triangle $b \cos A = a \cos B$, shew that the triangle is isosceles.

9. If two sides of a triangle be given, and the angle opposite to one of them; shew how to find the other side and the other angles.

Ex. The sides are 1 foot and $\sqrt{2}$ feet respectively, and the angle opposite to the shorter side is 30^0.

PREVIOUS EXAMINATION. *June*, 1882.

1. DEFINE the unit of circular measure. Calling this angle the radian, find how many radians there are in a right angle.

If the circumference of a circle be divided into five parts in arithmetical progression, the greatest part being six times the least, express in radians the angle each subtends at the centre.

2. Define the sine of an angle, wording your definition so as to include angles of any magnitude.

Prove that $\sin(90 + A) = \cos A$,

and $\cos(90 + A) = -\sin A$,

and by means of these deduce the formulæ

$$\sin(180 + A) = -\sin A, \ \cos(180 + A) = -\cos A.$$

3. Prove the formulæ

(1) $\cot^2 A = \operatorname{cosec}^2 A - 1$,

(2) $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A$.

Verify (2) when $A = 30^0$.

4. Shew that all angles satisfying the equation $\cos\theta = \cos\alpha$ are included in the formula $\theta = 2n\pi \pm \alpha$.

Solve completely the equation $2\cos^3\theta + \sin^2\theta - 1 = 0$.

5. Prove that $\sin(A+B) = \sin A\cos B + \cos A\sin B$, and deduce the expression for $\cos(A+B)$.

Shew that $\sin A\cos(B+C) - \sin B\cos(A+C) = \sin(A-B)\cos C$.

6. Prove that

$$\cos\frac{A}{2} + \sin\frac{A}{2} = \pm\sqrt{1+\sin A},$$

$$\cos\frac{A}{2} - \sin\frac{A}{2} = \pm\sqrt{1-\sin A},$$

and determine the signs of the ambiguities when A lies between 450^0 and 630^0.

7. Establish the identities

(1) $1 + \cos A + \sin A = \sqrt{2(1+\cos A)(1+\sin A)}$,

(2) $\operatorname{cosec} 2A = \dfrac{\operatorname{cosec}^2 A}{2\sqrt{\operatorname{cosec}^2 A - 1}}$,

(3) $\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} - \sin\dfrac{6\pi}{7} = 4\sin\dfrac{\pi}{7}\sin\dfrac{3\pi}{7}\sin\dfrac{5\pi}{7}$.

8. Shew that in any triangle (with the usual notation)

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

If $a=7$, $b=8$, $c=9$, shew that the length of the line joining the angular point B to the middle point of the opposite side is 7.

9. Solve completely the triangle whose sides are given in the preceding question, given log 2, log 6,

$L\tan 24^0.\,5' = 9{\cdot}6502809$, tabular difference for $60'' = {\cdot}0003390$,

$L\tan 36^0.\,41' = 9{\cdot}8721123$, tabular difference for $60'' = {\cdot}0002637$.

MATHEMATICAL TRIPOS. *January* 3, 1876.

1. DEFINE the cotangent and the cosecant of an angle, and trace the changes in the values of these functions as the angle increases from two to four right angles.

What values of θ satisfy the equation

$$\operatorname{cosec} 4\alpha - \operatorname{cosec} 4\theta = \cot 4\alpha - \cot 4\theta\,?$$

2. Prove the formulæ

(1) $\cos(A+B)=\cos A\cos B-\sin A\sin B$,

(2) $\cos B-\cos A=2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$.

Prove that if $A+B+C=180^\circ$, and if

$$\sin(2n+1)A\sin(B-C)+\sin(2n+1)B\sin(C-A)+\sin(2n+1)C\sin(A-B)=0,$$

n being an integer, then

$$\sin(n-1)A\sin(n+1)(B-C)+\sin(n-1)B\sin(n+1)(C-A)+\sin(n-1)C\sin(n+1)(A-B)=0.$$

3. Shew how to express the tangent of half an angle of a triangle in terms of the sides.

The sides are observed to be $a=5$, $b=4$, $c=6$, but it is known that there is a small error in the measurement of c: examine which angle can be determined with the greatest accuracy.

4. Shew how to find the height of a mountain by means of observations made at two given places in a horizontal plane.

On a plane which is inclined to the horizon at an angle of 45° is described a circle of known radius, and a post is placed at the highest point of the circle perpendicular to the plane; at one end of the horizontal diameter of the circle the tangent of the angular elevation of the post is $\sqrt{2}$. Find the length of the post.

5. If O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles of a triangle, and r, r_1, r_2, r_3 the radii of those circles, and R the radius of the circumscribing circle; shew that

$$4R=\frac{r}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{r_1}{\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}},$$

and that the areas of the triangles $O_1O_2O_3$, O_2O_3O, O_3OO_1, OO_1O_2 are to one another inversely as r, r_1, r_2, r_3.

6. From a point P perpendiculars PL, PM, PN are drawn to the sides of a triangle ABC: shew that twice the area of the triangle LMN is equal to

$$(R^2-d^2)\sin A\sin B\sin C,$$

where R is the radius of the circle circumscribing the triangle ABC, and d the distance of its centre from P.

MATHEMATICAL TRIPOS. 1869.

1. PROVE the formulæ:

(1) $\tan A = \tan(n\,.\,180^\circ + A)$, when n is a positive or negative integer.

(2) $\sin(A-B) = \sin A \cos B - \sin B \cos A$, where 180°, A, 90°, B, 0° are in descending order of magnitude.

(3) $2\sin\frac{A}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}$, where A is greater than 90° and less than 270°.

2. If β, γ be two different values of θ which satisfy the equation

$$\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta = \frac{1}{c},$$

then will

$$a\cos\frac{\beta+\gamma}{2} = b\sin\frac{\beta+\gamma}{2} = c\cos\frac{\beta-\gamma}{2}.$$

Having given

$$a^2\cos\alpha\cos\beta + a(\sin\alpha+\sin\beta) + 1 = 0,$$

$$a^2\cos\alpha\cos\gamma + a(\sin\alpha+\sin\gamma) + 1 = 0,$$

prove that

$$a^2\cos\beta\cos\gamma + a(\sin\beta+\sin\gamma) + 1 = 0,$$

and that

$$\cos\alpha + \cos\beta + \cos\gamma = \cos(\alpha+\beta+\gamma),$$

β, γ being unequal and less than π.

3. Investigate the expression for one side of a triangle in terms of the other sides and the angle included by them, and put it into a form proper for logarithmic computation.

Equilateral triangles DBC, $D'BC$ are described on the side BC of a triangle ABC; prove that $AD^2 + AD'^2 = a^2 + b^2 + c^2$, and express $\cos DAD'$ in terms of the sides a, b, c. Hence shew that if equilateral triangles be in like manner constructed on the other two sides, and the angles DAD', EBE', FCF' be denoted by α, β, γ,

$$\cos\alpha + \cos\beta + \cos\gamma = 0 = \cos 2\alpha + \cos 2\beta + \cos 2\gamma.$$

4. If A, B, C be angles of a triangle, and x, y, z any real quantities satisfying the equation

$$\frac{y\sin C - z\sin B}{x - y\cos C - z\cos B} = \frac{z\sin A - x\sin C}{y - z\cos A - x\cos C},$$

then will

$$\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}.$$

5. Given one side of a triangle and its angles, determine its area and the radii of the four circles which can be described touching the sides.

If a, b, c, d be the sides of a convex quadrilateral in which a circle can be inscribed, its area will be $\sqrt{abcd}\sin\omega$, 2ω being the sum of two opposite angles.

6. A gun is fired from a fort A and the intervals between seeing the flash and hearing the report at two stations B, C are t, t' respectively; D is

a point in the same straight line with BC at a known distance a from A: prove that if $BD=b$, and $CD=c$, the velocity of sound is

$$\left\{\frac{(b-c)\,(a^2-bc)}{bt'^2-ct^2}\right\}^{\frac{1}{2}}.$$

Examine the case where $a^2=bc$.

MATHEMATICAL TRIPOS. 1870.

1. Define a logarithm. Shew that in the common system of logarithms the integral part of any logarithm need not be tabulated. If a table of logarithms be calculated to any base, prove that they may be transformed to any other base by multiplication by a constant factor.

If a, b, c be in Geometrical Progression, and $\log_c a$, $\log_b c$, $\log_a b$ in Arithmetical Progression, the common difference of this progression is $\frac{3}{2}$.

2. Assuming that the formulæ

$$(1)\quad \sin(A+B)=\sin A\cos B+\cos A\sin B,$$
$$(2)\quad \cos(A+B)=\cos A\cos B-\sin A\sin B,$$

are true for angles less than a right angle, shew that they are also true for positive angles of any magnitude.

Prove the formulæ:

$$(1)\quad \cos 2A=2\cos^2 A-1=1-2\sin^2 A,$$
$$(2)\quad \cos^2 A+\sin^2 A\cos 2B=\cos^2 B+\sin^2 B\cos 2A,$$
$$(3)\quad \sin^2 A-\cos^2 A\cos 2B=\sin^2 B-\cos^2 B\cos 2A.$$

3. If $A+B+C$ be a multiple of 180°, then will

$$\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$$

If $A+B+C=(2m+1)\pi$ or $2m\pi+\frac{\pi}{2}$, then will

$$(\sin A+\cos A)\,(\sin B+\cos B)\,(\sin C+\cos C)$$
$$=2\sin A\sin B\sin C+2\cos A\cos B\cos C+1,$$

but if $A+B+C=2m\pi$ or $2m\pi-\frac{\pi}{2}$, then will

$$(\sin A+\cos A)\,(\sin B+\cos B)\,(\sin C+\cos C)$$
$$=2\sin A\sin B\sin C+2\cos A\cos B\cos C-1.$$

4. Shew how to solve a triangle in which are given the sides a, b and the angle A: and determine in what cases there are two solutions, one, or none.

If there be two triangles having the given parts, and if c_1, c_2 be their third sides, the distance between the centres of their circumscribing circles

$$=\frac{c_1\sim c_2}{2\sin A}.$$

5. If a, b, c be the sides of a triangle, and $2s=a+b+c$, then the area of the triangle $=\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$.

If S_1, S_2, S_3, S_4 be the areas of the four triangles whose sides are b, c, d; c, d, a; d, a, b; a, b, c respectively, prove that

$$\frac{S_1^2-S_2^2}{a^2-b^2}+\frac{S_3^2-S_4^2}{c^2-d^2}=\frac{S_2^2-S_3^2}{b^2-c^2}+\frac{S_4^2-S_1^2}{d^2-a^2}.$$

6. Find the radius of the circle which touches one side of a triangle and the other two produced.

If r, r' be the radii of two circles, and if P, P' be the perimeters, A, A' the areas of the triangles whose sides touch them, prove that

$$(1) \quad PP'(r-r')-rP'^2=4r^2r',$$

$$(2) \quad A^2(r-r')^2-A'^2(r+r')^2=4r^3r'^3,$$

r, P, A, being greater respectively than r', P', A'.

MATHEMATICAL TRIPOS. 1871.

1. DEFINE a logarithm, and prove, if a, b, N be any three numbers,

$$\log_a b \log_b a=1, \text{ and } \log_a N=\frac{\log_b N}{\log_b a},$$

prove also that

$$\frac{\log_a \log_a N}{\sqrt{\log_a b}}-\frac{\log_b \log_b N}{\sqrt{\log_b a}}=\frac{\log_a \log_a b}{\sqrt{\log_a b}}=-\frac{\log_b \log_b a}{\sqrt{\log_b a}}.$$

2. Find a general expression representing all the angles which have a given sine.

If $\quad \sin(\theta+\alpha)=\sin(\phi+\alpha)=\sin\beta,$

and $\quad a\sin(\theta+\phi)+b\sin(\theta-\phi)=c,$

prove that, either

$$a\sin(2\alpha\pm 2\beta)=-c, \text{ or } a\sin 2\alpha\pm b\sin 2\beta=c.$$

3. Prove the formula

$$\sin(A-B)=\sin A\cos B-\cos A\sin B,$$

where A is greater than B, and each of them less than 90^0. Also assuming this formula, prove

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

If $\quad \sqrt{2}\cos A=\cos B+\cos^3 B,$

and $\quad \sqrt{2}\sin A=\sin B-\sin^3 B,$

prove that $\quad \pm\sin(B-A)=\cos 2B=\frac{1}{3}.$

If $\quad 4\cos(x-y)\cos(y-z)\cos(z-x)=1,$

prove that

$$1+12\cos 2(x-y)\cos 2(y-z)\cos 2(z-x)=4\cos 3(x-y)\cos 3(y-z)\cos 3(z-x).$$

4. Investigate a formula connecting any angle of a triangle with the sides of the triangle.

O is any point in the interior of an equilateral triangle ABC, prove that

$$\cos(BOC-60^0)=\frac{BO^2+CO^2-AO^2}{2BO\,.\,CO}.$$

5. Find the radius of the circle which is described about a given triangle.

If G be the centre of gravity of a triangle, and R_1, R_2, R_3 be the radii of the circles circumscribing BGC, CGA, AGB respectively: prove that

$$\frac{a^2(b^2-c^2)}{R_1^2}+\frac{b^2(c^2-a^2)}{R_2^2}+\frac{c^2(a^2-b^2)}{R_3^2}=0.$$

6. Find the lengths of the diagonals of a quadrilateral inscribed in a circle in terms of its sides.

If $ABCD$ be the quadrilateral, Δ its area, and if CC' be a chord parallel to BD, find the length of CC' in terms of the sides, and prove that the radius of the circle is

$$\frac{AC'.AC.BD}{4\Delta}.$$

Note. The centre of gravity of a triangle is the intersection of the lines drawn from the angles to bisect the opposite sides.

The following tables comprise all the natural sines, &c. and logarithms required for the solution of the examples in this book; but the student is recommended where practicable to use complete tables, as he will thus become more familiar with their use. He will however only be able to arrive at an approximate solution, without the use of proportional parts. In many of the earlier examples the natural sines, &c. have only been employed to four decimal places. The student must remember that when he wants to find a cosine, he must look for the sine of the complement, and similarly for the cotangents and cosecants.

TABLE I.

Natural Sines, &c.

Sines.		Sines.	
5°. 58′	·1039499	67°. 23′	·9230984
7°. 46′	·1351392	68°. 13′	·9285933
11°. 32′	·1999380	73°. 32′	·9589848
12°	·2079117	Tangents.	
16°. 26′	·2828995		
19°. 28′	·3332584	10°. 40′	·1883495
21°. 47′	·3710977	26°	·4877326
25°	·4226183	26°. 34′	·5000352
32°	·5299193	33°. 41′	·6664969
32°. 53′	·5429302	37°	·7535541
34°. 24′	·5649670	50°. 46′	1·2246658
36°. 53′	·6001876	52°. 50′	1·3190441
38°. 13′	·6186370	61°	1·8040478
39°	·6293204	67°. 23′	2·4003774
40°. 39′	·6514366	Secants.	
43°. 26′	·6875101		
45°. 35′	·7142691	26°	1·1126019
48°. 24′	·7477981	52°. 50′	1·6552575
51°	·7771460	66°. 48′	2·5384453
61°. 3′	·8750425	68°. 12′	2·6927480
65°	·9063078	73°. 45′	3·5736108

TABLE II.

LOGARITHMS OF NUMBERS.

Numbers.	Logarithms.	Numbers.	Logarithms.
1·068185	·0286466	2·41	·3820170
1·1	·0413927	3	·4771213
1·143716	·0583182	3·09017	·4899824
1·22239	·0872111	3·652281	·5625642
1·3	·1139434	3·727593	·5714286
1·39009	·1430443	4·013116	·6034817
1·4458	·1601082	4·03826	·6061942
1·590643	·2015726	4·23665	·6270227
1·62335	·2104122	4·251787	·6285714
1·69408	·2289333	4·45813	·6491525
2	·3010300	7	·8450980
2·07	·3159703	7·39148	·8687314
2·3	·3617278	7·90672	·8979965

TABLE III.

TABULAR LOGARITHMS OF THE TRIGONOMETRICAL RATIOS.

Sines.	Logarithms.	Tangents.	
6°. 22′. 45″	9·0457574	3°. 43′. 39″·03	8·8139084
17°. 11′. 12″·11	9·4705539	10°.53′. 36″	9·2843181
18°	9·4899824	11°. 16′. 10″	9·2994355
23°. 42′. 43″	9·6043801	11°. 44′. 29″·5	9·3177416
29°. 5′. 21″·52	9·6867903	34°. 6′. 24″	9·8307301
41°. 8′. 54″·6	9·8182344	57°. 22′. 26″·8	10·1937091
41°. 42′. 43″	9·8230636	57°. 54′	10·2025255
48°. 51′. 5″·94	9·8767989	63°. 47′. 35″·59	10·3078513
53°. 27′. 20″	9·9049296	71°. 33′. 54″	10·4771213
62°. 6′. 51″	9·9463950		
64°. 25′. 49″	9·9552358		
72°	9·9782063		

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